ELEMENTS OF MATHEMATICAL ASTRONOMY

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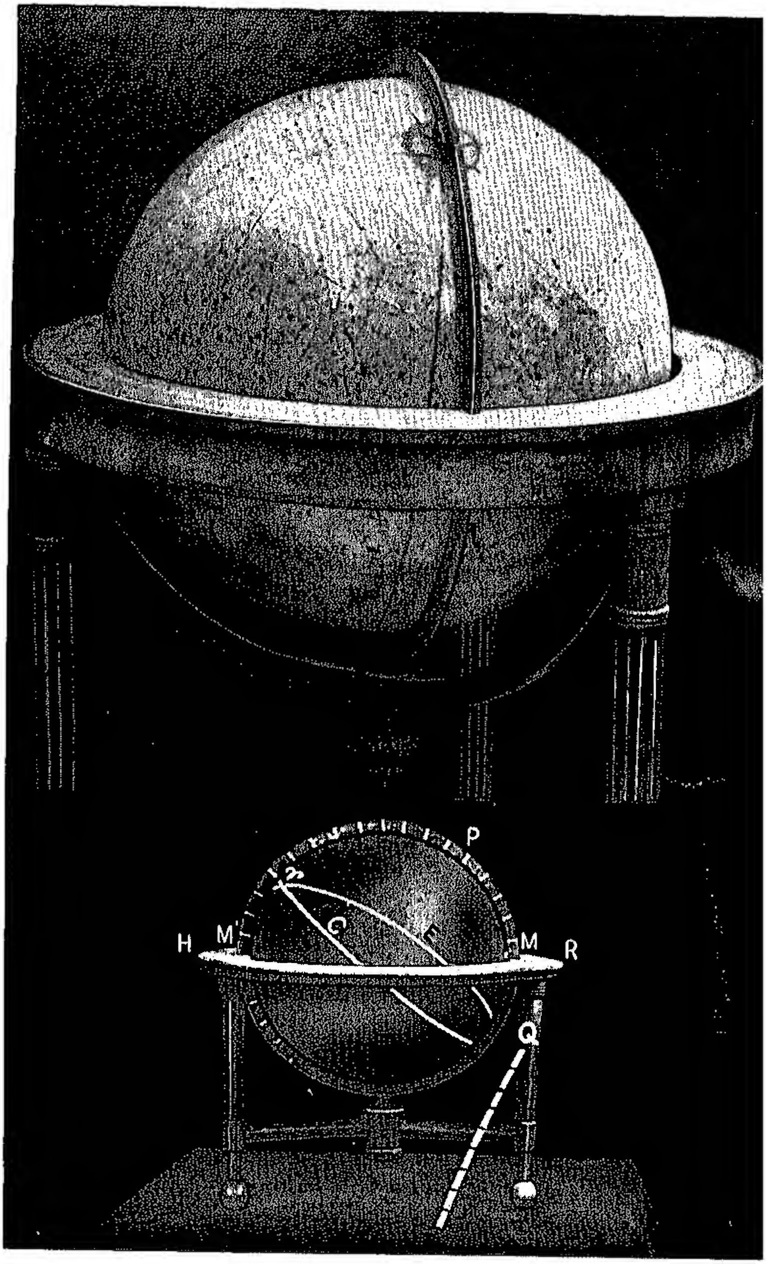

A 21-inch celestial globe and (lower figure) a small home-made globe showing the horizon *HR*, the equator *C*, the ecliptic *E*, and the meridian MM'.
See pp. 50–1 for explanation.

Elements of Mathematical Astronomy

With a Brief Exposition of Relativity

by

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ERRATA

It has been pointed out on p. 59 that an accumulation of small errors may occur, leading to a discrepancy of 1′ when 4-fig. tables are used. To ensure accuracy in the 4th figure in *all cases* it is safer to use 5-fig. tables but this is seldom necessary.

P. 23, the last paragraph should read as follows: "The radian measure of 14° 03′ is 0·2449, and multiplying this by 3442, the result is 843 miles approximately. Alternatively, 14° 03′ = 843′, which is the equivalent of 843 miles, and this is less than the departure."

P. 56, Example 1, line 5, *for* "θ" *read* "α".

P. 62, Example 10, line 3 from foot, *for* "cos X" *read* "X".

P. 63, Problem 8. See answer on p. 160 which should read, "51° 16′ and latitudes north of this".

P. 72. To make the diagram (Fig. 24) agree with the text, "06*s*" *should read* "04*s*".

P. 75, line 9. *For* "29" *read* "27", with corresponding differences in lines 11 and 12.

P. 76, last line. The h, as many would assume, refers to the hour angles when nautical twilight ends and begins, respectively.

P. 107, equation (47), *for* "abberration" *read* "aberration".

P. 115, line 12 from foot, *for* "og" *read* "log".

P. 115, lines 4 and 3 from foot, the letter τ should be a subscript.

P. 115, line 2 from foot, the first δ should be δ_1.

P. 121, line 18, "astronomica" *should read* "astronomical".

P. 129, line 18, after "dealt" *read* "elsewhere".

P. 129, line 19, after "gravity" *read* "(see *From Atoms to Stars*, p. 121, and also pp. 147–8 of this book)".

P. 136, line 14, *for* "less" *read* "smaller".

P. 143, line 25, for "1·27" read "1·127".

P. 149, equation (72), *after* "miles" *insert* "per second".

P. 154, line 6 from foot, *for* "–m" read "-m".

P. 159, Chapter II, Problem 11, *for* "eclipse" *read* "ecliptic".

P. 161, end of line 8 from foot, *read* "2a, and hence $2/r$ is of the same order as $1/a$ in (58)".

P. 184, line 5, *for* "west" *read* "east".

P. 198, line 14 from foot, *for* "cm." *read* "cm./sec.".

PREFACE

This book is intended primarily for those who have not the advantage of a teacher and are anxious to obtain a foundation of the principles of mathematical astronomy. Experience derived from a large amount of correspondence with students of this kind—usually amateur astronomers—has shown the necessity for a book of this nature to assist them over their initial difficulties. In many cases they find the usual text-book too advanced for their purpose and are often deterred from pursuing a course of studies in which they are intensely interested but which presents too many difficulties. Simple explanations for those who are thus handicapped will, it is hoped, prove of value.

The book follows the usual procedure adopted in mathematical treatises on astronomy but, in addition, it deals with a few problems of a special nature which are not included in the ordinary text-book. These are similar to, and in some cases identical with, problems proposed by members of the British Astronomical Association to which the author and other members contributed solutions. They will be found in the Appendices at the end of Part I, pp. 152–8.

Part II contains an elementary treatment of the subject of relativity. Some readers may find this portion too advanced, but it is an addition which has little bearing on the other part of the book, and the problems which arise and which are dealt with in the second part are all quite independent of the relativity theory. Those who can read it intelligently will find it a useful introduction to the relativity theory as expounded in more advanced works which would prove unreadable without some preliminary explanation such as Part II supplies.

One word of advice may not be out of place to amateur astronomers. At the present time it is difficult to obtain globes—terrestrial or celestial—and it is very important that a globe should be available for a proper understanding of a number of points explained in the text. A simple method for making one's own globe is described at the end of Chapter II, and although this apparatus is crude it will render invaluable assistance. The author has seen people with a good mathematical background utterly puzzled when they took up the subject of mathematical astronomy because the diagrams in the text did not show very clearly the three dimensional problems with which astronomy necessarily deals. The result was that they lost interest in the subject, which would not have happened if they had been in possession of a globe.

Readers should practise drawing circles—great and small—on the

globe which they can make for themselves. Of course some may globe, and if it is a celestial globe provided with a horizon, so m better. Great and small circles are marked on all globes—ter and celestial—and it is easy to verify some of the points ra Chapter I—for instance, that great circle sailing is an econc distance. The region of the pole should be selected for this purp it exemplifies the fact better than regions in mid-latitudes or n equator.

It will be a great advantage if readers have an elementary kno of plane and spherical trigonometry and it is assumed that they working acquaintance with the former at least. In spherical trigon the number of formulae which are essential for solving most problems that arise is small, and these are provided in the text, proofs. The methods of derivation, if required, can be found elementary book on the subject, but probably most readers who familiar with spherical trigonometry will be content to accept the out question.

It is too much to expect that many of those who read this bo possess computing machines. While these are a very great advant are practically indispensable for more advanced computations, log or a slide-rule are all that are required for the computations in or in the examples set at the end of each chapter. The methods a for logarithmic computations are abundantly illustrated, and the examples should not present any difficulty to those who foll illustrative exercises.

It is hoped that the simple treatment of the subject will fulfil it in assisting those who desire a background in mathematical astr

In conclusion, I have to acknowledge the kindness of Dr. J. G. Director of the Computing Section, British Astronomical Associat reading through the proofs and for some valuable suggestions.

M. Davi

1947 June.

PREFACE TO SECOND EDITION

In preparing a second edition of this book for the press the tunity has been taken to insert an errata page (p. 10) correctin minor errors that had crept into the text and improved diag replace some of the original illustrations.

I should like to thank Messrs. Macmillan & Co., Ltd., a University Tutorial Press, Ltd., for kindly allowing me to mak some of their diagrams as the basis of a few in this book.

M. David

1949 February.

PART I

ELEMENTS OF MATHEMATICAL ASTRON

Chapter I

THE EARTH

Before the reader is introduced to the elementary principles of mathematical astronomy it is essential that he should have a good working acquaintance with the earth—its motion, shape, dimensions, etc. To avoid unnecessary difficulties it will be assumed at first that the earth is a sphere—an assumption which is sufficiently accurate for most practical purposes—but when the greater accuracy required for certain special astronomical computations, such as eclipses, is demanded, the exact shape of the earth must be taken into consideration.

The earth rotates about a diameter which is called its *axis*, and this rotation can be simply illustrated by turning a sphere round on a rod passing through its centre. If one end of this rod, representing the axis, is pointed towards the pole star and the sphere is rotated from west to east, this affords a simple model of the earth. We can imagine that an observer is a very small speck somewhere on the surface of this sphere and sharing in its rotation. It is important to remember that when we are dealing with the earth the observer is assumed to be on the *surface* of the sphere, whereas in dealing with the heavenly bodies he is assumed to be *inside* the sphere, but this will be more fully explained in the next chapter.

Definitions

The axis meets the surface of the sphere in two points P and P' (Fig. 1), called the *poles*. If we imagine a plane drawn through the centre of the earth perpendicular to this axis, it will meet the surface of the earth in a circle which is called the *equator*. It is possible to draw an infinite number of planes at right angles to the axis, but only one of them will pass through the centre O. These other planes meet the surface in circles with smaller radii than that of the equator. Fig. 1 shows one of these circles and also the equator QQ'.

The section of the surface of a sphere by a plane is called a *great circle* if the plane passes through the centre of the sphere, and a *small circle* if the plane does not pass through the centre of the sphere. The equator is a great circle and the other circle shown in the figure is a small circle. The equator is not the only great circle which can be drawn

through any point on a sphere, nor are the small circles equator the only small circles which can be drawn th It is possible to draw an infinite number of each through surface of a sphere. Fig. 3 shows eight great circles joi another great circle midway between the poles—the equ small circles parallel to the equator.

Let c be any point on the surface of the earth and q parallel to the equator and passing through c (Fig. 1). A c and the axis PP' will meet the equator in C. If O and o

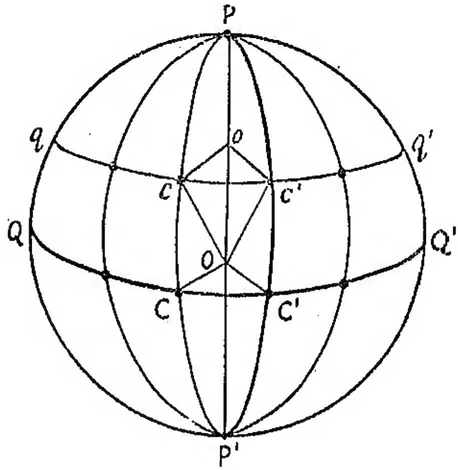

FIG. 1
The terrestrial sphere.

of the earth and of the small circle respectively, then parallel. The angle cOC between Oc and the plane o called the *latitude* of c, and will be denoted by ϕ. It is

of all places which lie on the small circle qq'. The *cola* complement of the latitude, so that colatitude $= 90^\circ$ lat

Let OC and oc be denoted by R and r respectively. lines is perpendicular to PP' and they are also paralle

Since $oc = Oc \sin \widehat{oOc} = Oc \cos \widehat{COc}$, it follows that

$$r = R \cos \phi \quad .. \quad ..$$

This formula applies to all the circles whose pla to the equatorial plane; such circles are called *parallel* ϕ is 0°, that is, if the latitude is that of the equator, (1) re which is otherwise obvious because in this case the pa

coincides with the equator. If $\phi = 90°$, $r = 0$, in other words, at either pole the parallel of latitude becomes a point.

Any plane drawn through the axis is called a *meridian plane.* The semicircle PcP' drawn through c is called the *meridian* of c. Similarly the semicircle $Pc'P'$ drawn through any other point c' is called the meridian of c'. It is convenient to have some meridian as the standard from which other meridians can be reckoned, and the meridian of Greenwich has been chosen for this purpose.

Suppose PcP' is the meridian of Greenwich and $Pc'P'$ is any other meridian. The arc CC' of the equator, intercepted between these two meridians, is the *longitude* of c'. Longitude is reckoned from 0° on the meridian of Greenwich eastward to 180°E. and westward to 180°W.

The angles coc' and COC' are equal, and if each of them be denoted by θ, then the arc $cc' = r\theta$, and the arc $CC' = R\theta$. Hence the arc cc' = the arc CC' multiplied by r/R. Since $r/R = \cos\phi$, it follows that

$$\text{arc } cc' = \text{arc } CC' \times \cos\phi \quad \ldots \quad \ldots \quad \ldots \quad (2)$$

As the earth rotates from W. to E. through 360° in 24 hours, or 15° per hour, different stars will cross the observer's meridian, or, to be more correct, the observer's meridian will be carried round so that it travels from west to east across different stars. In many problems in astronomy it is more convenient to assume that the stars are moving round the centre of the earth than that the earth is rotating. When we speak of a heavenly body rising or setting, which we shall frequently, it must be remembered that it is really the earth's rotation which is responsible for this phenomenon, but in spherical astronomy there is usually an advantage in dealing with the subject on the hypothesis of a fixed earth and moving stars.

Demonstrating the Earth's Rotation

A number of arguments can be brought forward in favour of the earth's rotation. If the stars revolved around the earth the velocities of those which are far off from us would be incredibly large. In addition, there would necessarily be some arrangement by which stars far away and those comparatively close would accomplish their diurnal motions in exactly the same time—a view which is utterly untenable unless the stars had a rigid physical connection. Other equally valid arguments can be adduced, but these depend, like those just referred to, on probabilities. Proofs by direct experiment are more convincing.

In Fig. 2 let OA be the radius of a sphere which is supposed to be rotating in the direction shown by the arrow head. Let B be a point on OA produced and let a body be dropped from B and fall towards

the centre O of the sphere. By a well-known principle in dynamics the body will not fall in a direct line towards O but will acquire a velocity in the direction BB' so that its actual path will be along the curve BC.

The sphere can be taken to represent the earth rotating in the direction of the arrow. Let B represent the top of a high tower and B' the position to which the top of the tower has moved in the interval during which the body would have fallen to the earth. Obviously the velocity of B exceeds that of the point A on the earth's surface. The result is that the body, moving horizontally with the same velocity as B, will not strike the earth at A' but at C a little to the side of A towards which the earth is rotating. The argument is applicable primarily to equatorial

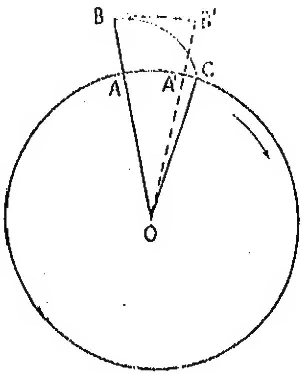

Fig. 2
Demonstrating the rotation of the earth by a body falling from a tower.

regions, but it applies also to any latitudes except that of the poles, with certain modifications with which we need not deal.

Experiments that have been carried out to test the above theoretical considerations show that bodies falling from a high tower do actually strike the surface of the earth a little to the east of the foot of the tower. The only explanation that can be offered for this deviation from the direction of the plumb line is that the earth is rotating from west to east. The experiment is, however, difficult to carry out, as the deviation is very small—about one-seventh of an inch for a fall of 100 feet at the equator.

A simpler proof of the earth's rotation is afforded by Foucault's pendulum experiment which readers may have seen for themselves in the South Kensington Museum or elsewhere. It can be imitated on a very small scale by means of a globe, terrestrial or celestial, and this imitation of the experiment is worth trying. The fundamental principle in Foucault's experiment depends on the tendency of a heavy body suspended by a cord and swinging backwards and forwards to maintain

the plane of its swing when the point of suspension is rotated. To verify this, construct a small pendulum swinging on a thread and suspended as shown in Fig. 3 (a).

Hold the base of the support for the pendulum on the surface of a globe as near the pole as possible and start the pendulum swinging. Now ask someone to rotate the globe slowly and notice that the plane in which the pendulum is swinging will not move round with the globe but will pass through different points on its surface.

If the experiment is repeated at the equator it will be found that the plane of the swinging pendulum is simply carried round, partaking of the general motion of rotation round the axis parallel to the plane of the

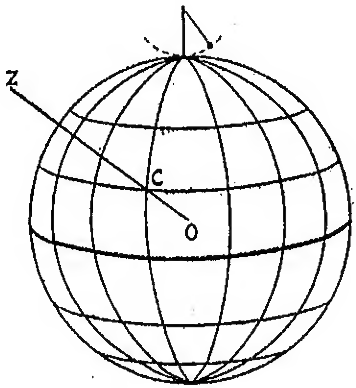

FIG. 3 (a)
Foucault's pendulum in polar regions showing the earth's rotation.

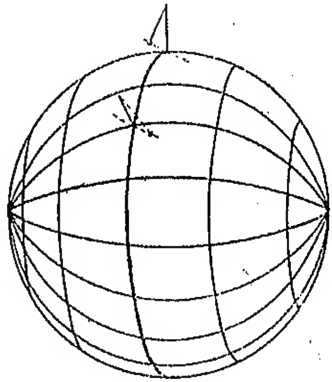

FIG. 3 (b)
At the equator Foucault's pendulum does not show the earth's rotation.

equator. Whatever be the plane of the swinging pendulum at the equator, there is nothing to produce a disturbance in this plane relative to a horizontal plane in the neighbourhood of the pendulum (see Fig. 3 (b)).

At places intermediate between the equator and the poles the conditions will differ from those at the equator but will be partly the same as those at the pole. At the pole the complete revolution of the plane of swing relative to the earth takes place in 24 hours, and at the equator the time is infinite because in equatorial regions no revolution of this plane relative to the earth takes place. At the place with latitude ϕ the time of a complete revolution is 24 cosec ϕ hours. Although a pendulum suspended in the manner described will not oscillate long enough to make a complete circuit, it will do so during a sufficient time to enable us not only to verify the earth's rotation, but also to use the above expression for the time of a complete revolution. We can make our computations from the arc of revolution in a certain time, and then by a simple proportion ascertain what period would correspond to a revolution of

360°. The actual results agree well with the theoretical results, and Foucault's pendulum affords an excellent proof of the earth's rotation.

Other direct proofs of the earth's rotation are available, with which we need not deal, and we shall proceed to consider some other problems connected with the earth.

Units of Measurement

A nautical mile is the distance on the earth's surface between two points A and B which subtend an angle AOB of one minute of arc at

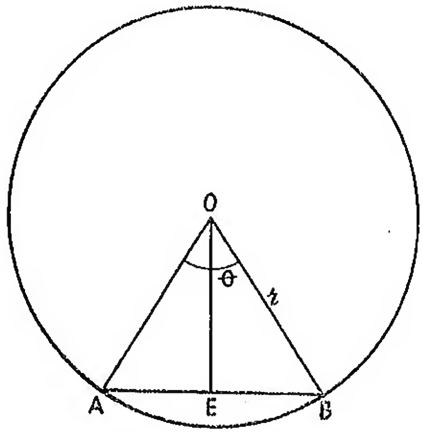

FIG. 4 (a)
Used to prove that great circle sailing is an economy in space. See text for explanation.

the earth's centre. Since the circumference of a circle contains 360°, which is 21,600 minutes of arc, a nautical mile is obtained by dividing the circumference of the earth by 21,600. If we take the earth's equatorial radius as 3963·35 English miles its circumference is 2π times this, or 24,902·44 English miles, and hence the length of an arc of one minute on the earth's surface is 1·15289 English miles or 6087 feet. Owing to the fact that the earth is not a sphere the length of the nautical mile varies for different latitudes, being greater in polar than in equatorial regions. Strictly speaking, a *nautical mile* is the length of a minute of arc of the *meridian* and a *geographical mile* is the length of a minute of arc measured on the *equator*. On the false assumption that the earth is a sphere there is no difference between a nautical and a geographical mile, and in practice this difference is usually ignored and a nautical mile is taken as 6080 feet. One degree on a great circle on the earth corresponds to 60 nautical miles.

An English or a statute mile is 5280 feet and hence 38 statute miles contain 200,640 feet. A nautical mile is taken as 6080 feet and hence 33 nautical miles contain 200,640 feet also, so we have the relation,

38 statute miles = 33 nautical miles.

Referring again to Fig. 1, the number of nautical miles in the arc CC' is simply the number of minutes in the equatorial arc CC'. The length of the arc cc', measured in nautical miles, is found by multiplying

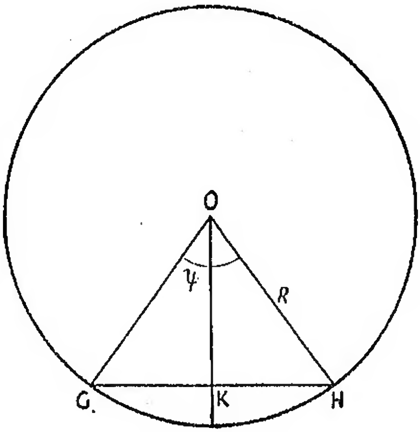

FIG. 4 (b)
Used to prove that great circle sailing is an economy in space. See text for explanation.

the difference in longitude between c and c', expressed in minutes of arc, by the cosine of the latitude, which is obvious from (2). This length is called the *departure* and is always expressed in nautical miles.

If a ship sails from c to c' and follows the small circle, sailing all the time parallel to the equator, the distance is greater than if a great circle passing through cc' had been followed. At first this may not seem very obvious, but the following considerations will show that great circle sailing is economical so far as saving distances is concerned.

In Fig. 4 (a) O is the centre of a circle of radius r and A and B are any two points on its circumference. Draw OE at right angles to the chord AB and let the angle AOB be θ. Since $AE = EB = r \sin \frac{1}{2} \theta$, it follows that $AB = 2r \sin \frac{1}{2} \theta$.

Now suppose we have another circle of radius R and that two radii of this circle are drawn meeting its circumference in the points G and H, and that GH is $2r \sin \frac{1}{2} \theta$. What is the angle GOH? (See Fig. 4 (b).)

From O draw OK at right angles to GH. In the right-angled triangle OGK, $GK = GO \sin \widehat{GOK}$, or $r \sin \frac{1}{2}\theta = R \sin \frac{1}{2}\widehat{GOH}$. Hence it is possible to find the value of the angle $\widehat{GOH}$, which will be denoted by ψ, from the expression

$$\sin \tfrac{1}{2}\psi = \frac{r}{R} \sin \tfrac{1}{2}\theta = \cos\phi \sin \tfrac{1}{2}\theta \quad \ldots \quad \ldots \quad (3)$$

In Fig. 1, θ is the angle coc', ψ is the angle cOc', cc' is AB and CC' is GH.

The application of (3) will be shown later in an example, from which it will be made clear that the great circle sailing always economizes in distance. The problem has been restricted to places in the same latitude and has been dealt with by plane trigonometry, but it will be shown later how general cases are solved.

The Visible Horizon

If an observer O is situated above the surface of the earth, the length of the tangents OT and OT' will limit his range of vision (Fig. 5). A small circle TT' formed by the revolution of the point T or T' about the diameter AB will constitute the *visible horizon* or *offing*, and this will depend on the height of the observer above the horizon.

From the elementary properties of the sphere we have the relation

$$BO \cdot OA = OT^2$$

If OA be denoted by h and OT by d, then, r being the earth's radius,

$$(2r + h)h = d^2, \text{ from which}$$

$$h^2 + 2rh - d^2 = 0$$

Solving this quadratic for h, we find

$$h = -r \pm \sqrt{(r^2 + d^2)}$$

The expression under the radical can be written in the form

$$\sqrt{(r^2\,(1 + d^2/r^2)\,)} = r\sqrt{(1 + d^2/r^2)}.$$

Expanding $\sqrt{(1 + d^2/r^2)}$, we find the above becomes

$$r\,(1 + d^2/2r^2 - d^4/8r^4 + \ldots)$$

$$\text{Hence } h = -r \pm r\,(1 + d^2/2r^2 - d^4/8r^4 + \ldots) = d^2/2r - d^4/8r^3 \quad (4)$$

Ignoring the second term, the above reduces to

$$h = d^2/2r \text{ as a first approximation} \quad \ldots \quad \ldots \quad \ldots \quad (5)$$

Knowing the distance d of an object which is just visible on the horizon, the eye of the observer being supposed to be on the horizon, the height of the object can be found from (5).

It is more useful to be able to ascertain the distance of an object just visible on the horizon if the height h of the observer is given. This is found from

$$d = \sqrt{(2hr)} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (6)$$

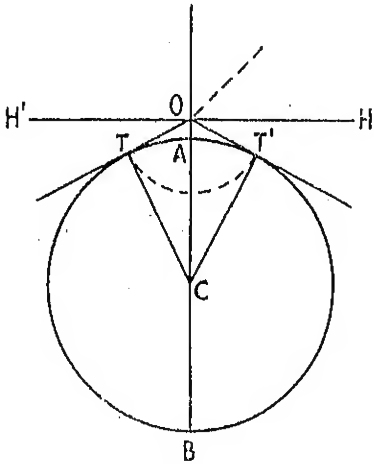

FIG. 5
The visible horizon.

If d and r are measured in nautical miles and h in feet, then since a nautical mile is 6080 feet and r is 3442 miles at the equator (but this value can be used for all latitudes),

$$d = \sqrt{(6884h/6080)} = \sqrt{(1 \cdot 132h)} = 1 \cdot 064 \sqrt{h} \quad \ldots \quad (7)$$

Hence if h is given in feet d is easily found in nautical miles.

If statute miles are required the factor 1·225 should be substituted for 1·064 in (7). Hence

$$d = 1 \cdot 225 \sqrt{h} \quad \ldots \quad \ldots \quad \ldots \quad (7\text{A})$$

for statute miles.

If d is given and h is required it is only necessary to square each side of the two formulae (7) and to simplify the results. These give

$$h = 0 \cdot 883d^2 \text{ for nautical miles} \quad \ldots \quad \ldots \quad \ldots \quad (8)$$
$$h = 0 \cdot 666d^2 \text{ for statute miles} \quad \ldots \quad \ldots \quad \ldots \quad (8\text{A})$$

To illustrate the principles considered in this chapter the following examples are worked out fully. Four-figure tables will suffice in all cases.

Example 1

Two places at latitude 50° have longitudes 5° E. and 10° W. What is the difference in their longitudes and what is the arc of the small circle between them (the departure)?

Since one place is east of the meridian of Greenwich and the other is west, their longitudes must be *added* to find the difference of their longitudes. This is 15°.

To solve the second part notice that each degree of longitude corresponds to 60 nautical miles and hence at the equator a difference of 15° corresponds to 900 miles. (In future "miles" will mean nautical miles unless otherwise stated.) The great circle arc CC' is, therefore, 900 miles.

log 900	2·9542
log cos 50°	$\bar{1}$·8081
log arc cc'	2·7623
arc cc'	578·4 miles

Example 2

A small circle parallel to the equator is drawn in latitude 60°. What is its radius? (The earth's equatorial radius can be taken as 3442 nautical miles.)

From (1) we have $r = R \cos \phi$.

log R	3·5369
log cos 60°	$\bar{1}$·6990
log r	3·2359
r	1721 miles

Example 3

A Foucault pendulum is oscillating in the latitude of Greenwich (about $51\frac{1}{2}$° N.). If you observe it for 20 minutes through what arc would it have appeared to rotate relative to the earth? In what direction, viewed from above, would this rotation appear to take place?

The time to complete a rotation is 24 cosec $51\frac{1}{2}$°. The computation is as follows:

log 24	1·3802
log cosec $51\frac{1}{2}^\circ$	0·1065
log time	1·4867
time	30·74 hours

The time of a complete rotation through 360° is 30·74 hours and hence in 20 minutes the arc described is

$$\frac{20 \times 360^\circ}{30{\cdot}74 \times 60} = 3^\circ{\cdot}9$$

The direction in which the plane of the pendulum appears to rotate relative to the earth can be easily determined by considering the special case at the north pole. If we imagine someone looking down on the north pole from above, the rotation of the earth is in a direction opposite to that of the hands of a watch. Hence the apparent movement of the plane of the pendulum is clockwise. The same argument applies to all latitudes between the north pole and the equator. In the southern hemisphere the opposite effect prevails.

Example 4

A ship steams along the parallel of latitude 41° 12′ from a place in longitude 40° 18′ W. to a place in longitude 21° 36′ W. Find the departure between the two points and also find the distance if great circle sailing is adopted.

The difference of longitude is 18° 42′ = 1122′.

log 1122	3·0500
log cos 41° 12′	$\bar{1}$·8765
log dep.	2·9265
dep.	844·3 miles

Referring to (3), $\theta = 18^\circ\ 42'$, $\frac{1}{2}\theta = 9^\circ\ 21'$.

log cos ϕ	$\bar{1}$·8765
log sin $\frac{1}{2}\theta$	$\bar{1}$·2108
log sin $\frac{1}{2}\psi$	$\bar{1}$·0873
$\frac{1}{2}\psi$	7° 01′·5
ψ	14° 03′

The radian measure of 14° 03′ is 0·2449, and multiplying this by 3442, 14° 03′ = 843′ = 843 miles. Alternatively, the result is 843 miles approximately, which is less than the departure.

Example 5

A lighthouse is visible at a distance of 24 miles, the eye of the observer being close to the level of the water. What is the height of the lighthouse?

Substituting 24 for d in (8) the result is as follows:

log 24	1·3802
2 log 24	2·7604
log 0·883	$\bar{1}$·9460
log h	2·7064
h	509 feet

Example 6

What error is committed by ignoring the second term in (4)?

It is necessary to find the value of $d^4/8r^3$ (see (4)), and using four-figure logarithms we proceed as follows:

log 24	1·3802
4 log 24	5·5208
log r	3·5369
3 log r	10·6107
log 8	0·9031
log $8r^3$	11·5138
log $d^4/8r^3$	$\bar{6}$·0070
$d^4/8r^3$	0·00000102

From (4) it is seen that the height is less than that given by (8) by 0·00000102 mile, that is by about 0·06 inch. This shows that the neglect of the second term in (4) is of no practical importance.

Example 7

What would be the height of the lighthouse in the above example if statute miles were used?

The logarithm of 0·666 is $\bar{1}$·8235, and substituting this for $\bar{1}$·9460 it is easily found that log h = 2·5839, and hence h = 384 ft. to the nearest foot.

Problems

1. Find the difference in latitude between two places A and B, given that their latitudes are:

(a) A, 35° N.; B, 52° N. (b) A, 40° 12′ S.; B, 37° 18′ N.; (c) A, 90° N.; B, 90° S.

2. Find the difference in longitude between two places A and B, given that their longitudes are:
(a) A, 25° 13′ E.; B, 72° 10′ E.; (b) A, 28° 10′ W.; B, 16° 23′ E.; (c) A, 110° 23′ E.; B, 72° W.

3. A ship steams eastward along the parallel of latitude between two places A and B at a speed of 20 knots. If A is 38° N. and 47° W., and the ship arrives at B in $18\frac{1}{2}$ hours, what is the longitude of B?

4. Find the great circle distance between A and B in 3.

5. Find the height of a mountain which is just visible from sea-level at a distance of 70 miles.

6. How long would Foucault's pendulum require to turn through 2° in latitude 40?

Note on the Signs of Trigonometrical Functions in Different Quadrants

Probably most readers are conversant with this matter, but for the sake of those who are doubtful the following points should be noticed.

In the first quadrant, that is, where the angles lie between 0° and 90°, the sine, cosine and tangent are positive. In the second quadrant, from 90° to 180°, the sine is positive, the cosine and tangent being negative. In the third quadrant, from 180° to 270°, the sine and cosine are negative and the tangent is positive. In the fourth quadrant, from 270° to 360°, the sine and tangent are negative and the cosine positive. The cosecant, secant, and cotangent have always the same sign as the sine, cosine, and tangent, respectively.

When the signs of two of the functions of an angle are known, its quadrant can always be determined by the above rules. This is shown in some of the examples in Chapter III.

Chapter II

THE CELESTIAL SPHERE

It is unfortunate that we have not got a planetarium in the British Isles. America and several other countries have a number of them and they are extremely useful when popular lectures on astronomy are delivered to an audience consisting for the most part of amateurs. As we have not a planetarium we must try to visualize the motions of the heavenly bodies without the aid of such a valuable apparatus.

In Chapter I we regarded the earth as a sphere, the observer being a very small object anywhere that he chose on its surface. When we deal with the stars it will be necessary to modify this view and to imagine that the inside of the sphere is studded with stars and that the observer is inside the sphere—at its centre—so that he is looking at a hollow spherical dome. To show this more clearly Fig. 6 should be studied very carefully.

O is the centre of a sphere which may have any diameter—about a hundred feet in the case of some planetaria but a matter of two feet or less in the case of the usual celestial globes. The line drawn from *O* to the stars *A*, *B*, *C*, *D*, etc., will intersect the surface of this sphere in points *a*, *b*, *c*, *d*, etc., and hence, if we could imagine the vault of the heavens reduced to a very small model, the sphere would represent this vault, the observer at *O*, and the whole earth itself having shrunk to a mere speck. Although the stars *A*, *B*, *C*, etc., are represented by the points *a*, *b*, *c*, etc., on the surface of the sphere, this does not imply that the stars lie on a sphere. They are at various distances from the earth—from a little over 4 light-years to millions of light-years—but for certain computations it is convenient to represent them as situated on the surface of the sphere.

If the earth shrinks to a mere point *O* the same cannot be said about the horizon. We have already explained the meaning of the *visible horizon* on p. 20, but a definition of the word *horizon* will assist in understanding certain methods of computation which follow.

Referring to Fig. 3, if *c* is an observer on the earth's surface, the prolongation of the radius *Oc* defines the direction of the *zenith* *Z* at *c*. The point diametrically opposite to the zenith is known as the *nadir*. A plumb-line held at *c* can be regarded as determining this direction since the plumb-line points from *c* to *O*. A plane through *c* perpendicular

to the direction of the zenith, or nadir, is known as the *horizon*, and it may extend for any distance.

Suppose the axis of the earth points exactly to the pole star—a supposition which is not true, but we shall assume for the present that it is true as this supposition simplifies the explanation—it is obvious that an observer at the pole would see the pole star in his zenith. If he were at the equator the pole star would be seen on his horizon, and if he went south of the equator it would not be seen at all, the surface of the earth in southern equatorial regions intercepting the light from it.

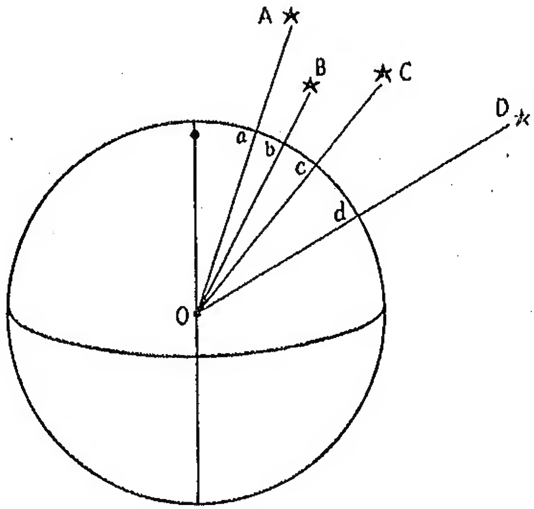

FIG. 6
The celestial sphere.

At latitudes between the equator and the pole the pole star would appear at various altitudes which would depend on the latitude.

The position which a star or any other heavenly body occupies on the celestial sphere can be referred to the observer's horizon and his meridian. Let Fig. 7 represent a portion of the observer's celestial sphere and Z his zenith at any instant. Any plane through the zenith and the observer O will be a vertical plane and its intersection with the celestial sphere is called a *vertical circle* or simply a *vertical*. One vertical in particular must be noted. Suppose the plane through Z and O passes through the east and west points E and W and is perpendicular to the observer's meridian; in this case the vertical circle obtained by its intersection with the celestial sphere is known as the *prime vertical*.

Let S be a star (Fig. 7), Z the zenith, HR the plane of the horizon, and ZST the vertical through the star meeting the horizon in T. R and H are the north and south points, and the great circle HZR is the meridian

of the observer. We have already defined the meridian (terrestrial) as the semicircle drawn through the observer and the earth's axis. The celestial meridian is simply the great circle in which the terrestrial meridian meets the celestial sphere, so if we could imagine the earth's centre at O, and the observer's terrestrial meridian at any time extended to intersect the sphere in the great circle HZR, this is the celestial meridian. The observer's celestial meridian always passes through his zenith.

The *azimuth* of the star S is the arc RT of the horizon measured from the north point R to the vertical of the star. It can also be defined as the spherical angle TZR which the star's vertical makes with the

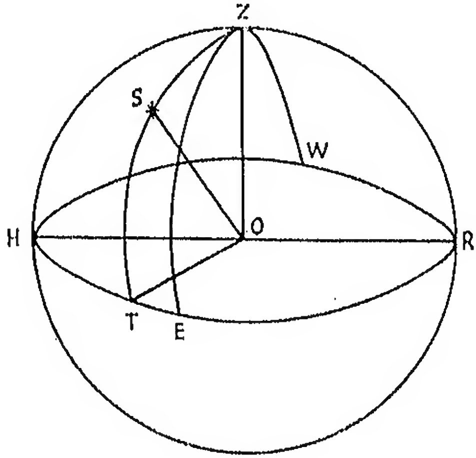

Fig. 7
The celestial sphere showing the observer's horizon.

meridian. The manner in which azimuth is measured must be clearly understood, especially as the modern method differs from that given in the older text-books. Azimuth is measured from the north up to 180° only, eastward or westward. Thus, if the arc RT or, what is the same thing, the spherical angle TZR, is less than 180°, the azimuth of the star is given as so many degrees east. If this angle exceeds 180°, say it is 200°, the azimuth is not said to be 200° east but 160° (360°–200°) west.

For an observer in the southern hemisphere the azimuth is measured from the south point up to 180°, eastward or westward.

The angle TOS is called the *altitude* of the star and is the star's angular distance from the horizon, measured along a vertical. The angle ZOS is the *zenith distance* of the star and is the complement of the altitude, so that a star's zenith distance is 90°– the star's altitude.

When the azimuth and altitude (or zenith distance) of a star are given for any instant, its position is defined uniquely for the particular latitude, and there is no difficulty in locating it provided one is equipped with an instrument for measuring azimuths and altitudes.

It should be pointed out that at present we are dealing only with the bodies very far away from the earth—so far that the same celestial sphere serves to show the apparent positions of the stars, even after many scores of years. Those who possess an old celestial globe will find that it is practically as good as a modern one for this purpose because, though all the stars are moving, yet, owing to their enormous distances from us, these movements are generally inappreciable even after a century, when reduced to the scale of a celestial globe.

The same thing does not apply to the sun, moon, planets, satellites and comets—members of the solar system. All these are relatively close to us and hence their movements in a short period are appreciable. We are not concerned with these at the moment and shall confine our attention to the stars.

The reader is strongly advised to set his globe, even a home-made one if he has not got a proper celestial globe, for different latitudes and so visualize the actual conditions under which observations are made. This is of special importance in connection with the next point with which we shall deal.

The Altitude of the Pole is Equal to the Latitude of the Place

In Fig. 8 let the sphere represent the earth and let C be the position of an observer on it, PCP being his meridian and the horizontal circle the equator. The prolongation of the line OC from the centre of the earth to C is in the direction of the observer's zenith. Let the arrow at P, the north terrestrial pole, point to the pole star which is supposed to be at an infinite distance. From C draw CS parallel to OP and let the angle SCZ be denoted by z. Because CS is parallel to OP the angle POC is equal to the angle SCZ and is therefore z. Notice that although there may be a distance of thousands of miles between the lines OP and CS, yet the pole star is in practically the same direction as seen from O and C because such a small distance as thousands of miles compared with the enormous distance of the pole star is insignificant.

The angle COC' is the latitude ϕ of the place, and since the sum of the angles COC' and POC is $90°$, it follows that

$$\phi + z = 90°$$

But if a is the altitude of the pole star, we know from what was previously stated that $z = 90° - a$, and hence

$$\phi + 90° - a = 90°, \text{ from which}$$
$$\phi = a$$

This simple relation between the latitude of the place and the altitude of the pole star is very important and will be frequently used in the solution of various problems.

It may now be pointed out that the pole star is about a degree from the north pole, which is the point on the celestial sphere to which the earth's axis points. For this reason, instead of speaking of the pole star

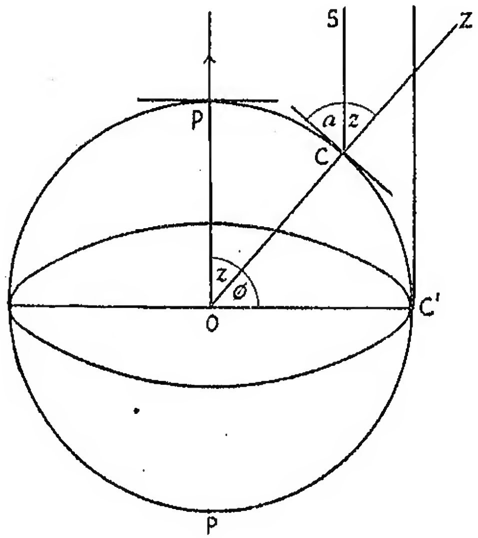

Fig. 8

Proof that the altitude of the celestial pole is equal to the latitude of the observer.

it is more correct to speak of the north pole of the heavens. Hence the above relation is accurately described as follows:

altitude of the celestial pole = latitude of observer

This applies to the south celestial pole also. There is no bright star as close to the south celestial pole as the pole star is to the north celestial pole, and we must imagine a point corresponding to the south celestial pole. Such a point would be on the horizon of an observer at the equator, at an altitude of 50° for an observer in latitude 50° S., and in the zenith, or at an altitude of 90° for an observer at the south pole of the earth, and so on.

We have seen that the earth rotates on its axis from west to east in 24 hours, but the same effect would be produced if we imagine that the earth is fixed (as the ancient astronomers thought) and that the whole sphere of the stars is turning round the centre of the earth from east to west. Hence it is necessary for the reader to imagine that he is inside the celestial sphere and that it is turning round him from *east to west*. When using a celestial globe make certain that this direction of revolution is observed.

Before proceeding to consider other means for defining the position of a heavenly body, the following experiments should be carried out on a celestial globe on which, if it is a home-made one, a few marks should be made to indicate stars scattered about in the celestial vault.

The horizon cannot, of course, be continued inside the sphere, but the reader can imagine that it is so continued and that he is situated on it at the centre of the celestial sphere. The celestial equator, which is the circle on the celestial sphere in which the plane of the terrestrial equator meets the latter, should be marked on the home-made globe, as the directions given on pp. 50–52 indicate. If possible, a rough scale showing about every 5 degrees on the meridian should be constructed: this will amply repay the labour involved, and the scales will be an advantage when the sphere is set for various latitudes. Although the horizon of the observer alters its direction in space as he moves over different latitudes, it would be inconvenient to alter the horizon of a celestial globe. It is simpler to maintain the horizon fixed and to alter the celestial sphere, just as it is more convenient to keep a fixed earth and to imagine that the heavenly bodies are moving round it.

The Apparent Movements of the Heavens for Various Latitudes

Place the poles on the horizon and notice the position of the equator. It will be seen that it is at right angles to the horizon, and if the globe is turned round, all the stars, whatever their positions may be, will move in circles which are perpendicular to the horizon. When the pole is on the horizon its altitude is 0°, and as the altitude of the pole is equal to the latitude of the place, the latitude is 0°, in other words, the observer is at the equator. The globe being thus set for the equator it is easy to see what happens there.

As both the north and south celestial poles are on the horizon, they are just visible at the equator, or perhaps it would be more correct to say that they would be visible if it were not for the effects of absorption of light by the atmosphere. Although this is considerably less in tropical countries than it is in the British Isles, nevertheless it would scarcely be correct to say that the portions of the sky representing the poles are visible at the equator. Assuming *ideal conditions*, however, we can say

that they are visible at the equator, and as further experiments will show, there is no other place on the earth where they are both visible.

Notice that all the circles described by the stars are divided into two equal portions by the horizon, and hence to an observer at the equator all stars will always be 12 hours above his horizon and 12 hours below it. The equator is unique in this respect as other experiments will show. The phenomena described above are shown in Fig. 9.

Now set the globe so that the equator corresponds with the horizon, either pole in this case being at an altitude of 90°. Since the altitude of

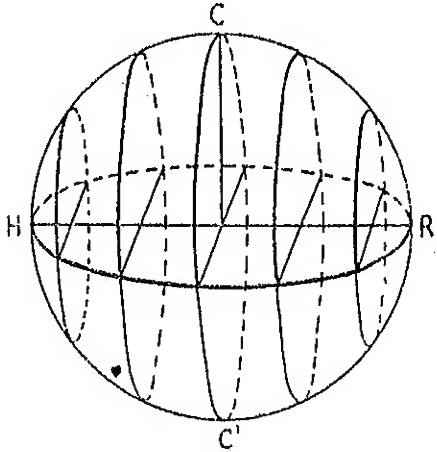

FIG. 9
The celestial sphere when the observer is at the equator.

the pole is the same as the latitude of the place, the latitude is 90°, or we are dealing with a place at the pole (for convenience we shall take it as the north pole). Turn the globe round in the usual manner—from east to west—and notice that no stars either rise above or set below the horizon. Those at the equator just skim the horizon, those south of it are below the horizon and so are invisible, while those north of it move in small circles parallel to the equator, neither rising nor setting. (See Fig. 10.) (By "equator" is meant, as readers will probably know, the celestial equator.) The above description gives a representation of what an observer at the pole would see, and is very different from the conditions under which an observer in equatorial regions sees the heavens.

Intermediate latitudes can be represented in a similar manner. Thus, suppose we want to know how the heavens appear to an observer

in our islands, set the globe so that the arc from the horizon *HR* to the north pole is about 52°. *Z* and *N* are respectively the zenith and nadir of the observer. On rotating the globe a state of affairs different from either of the others prevails. (See Fig. 11.)

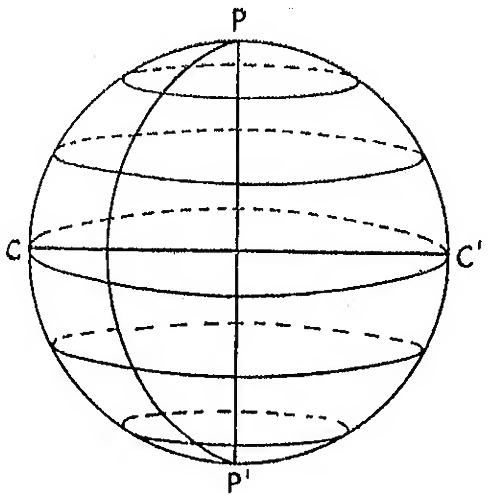

FIG. 10
The celestial sphere when the observer is at the pole.

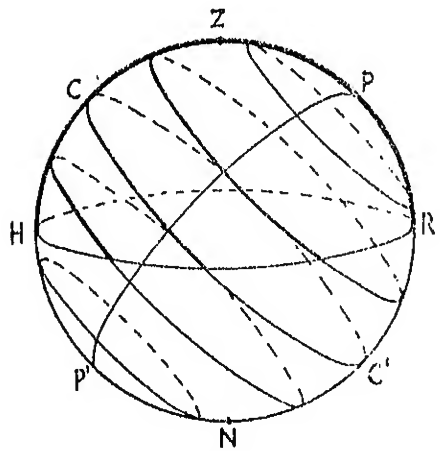

FIG. 11
The celestial sphere when the observer is at a latitude intermediate between the equator and the pole.

Circumpolar Stars

First of all, if some of the marks representing stars have been placed on the equator CC', which now represents the celestial equator, it will be seen that all such stars rise exactly in the east and set exactly in the west. This phenomenon takes place for other latitudes as can easily be verified, even for the equator, but it cannot be said to occur at either pole because in this case stars on the celestial equator just skim the horizon. Notice that stars fairly close to the pole do not set; they approach the horizon HR but do not go below it. Others further off from the pole move near the horizon, and others, if situated at the correct distance from the pole, just touch the horizon but do not move below it. Stars which neither rise nor set but move round and round the pole are known as *circumpolar stars*. Others a little further off from the pole than those that just touch the horizon rise and set but remain a very short time below the horizon. Others further off still remain a longer time below the horizon but spend most of their time above it, while those very far away from the pole and near the equator divide their time into nearly equal portions above and below the horizon, the former being the greater. Stars on the equator are 12 hours above and 12 hours below the horizon, and when we observe the stars south of the equator we shall find that they are less than 12 hours above, and more than 12 hours below, the horizon. These facts should be verified and the experiments will serve as a check—if only a rough check—on the results obtained later by the use of certain formulae.

In whatever latitude the globe is set, except that of the equator, it will be found that one celestial pole is above and the other below the horizon. Hence at no latitude, except that of the equator, is it possible to see all the stars in the celestial sphere. In higher latitudes some will be invisible in one or the other hemisphere.

The Ecliptic and the First Point of Aries

There is a great circle which must be drawn on the celestial sphere if the explanations which follow are to be properly understood. This circle can be drawn as follows and is shown in Fig. 12.

On the equator CC' take any point which should be marked ♈ and with a scale measure 90° eastward from ♈ along the equator to another point C'. On the great circle connecting C' with the north pole P measure $C'E'$ equal to $23\frac{1}{2}$°. From C' on the equator measure another arc C' ♎ equal to 90°. Continue round the equator and mark the point C 90° from ♎, and on the great circle connecting C with P', the south pole, measure CE equal to $23\frac{1}{2}$°. By means of a flexible strip of steel or brass

draw a great circle through the four points ♈ E' ♎ E. The great circle ♈ E' ♎ E around the sphere is the *ecliptic*, in which the sun always moves. The first point selected, which is one of the two points of intersection of the ecliptic and the equator, is very important because it is the zero point from which certain measurements are made. It is called *the first point of Aries*, and is denoted by the symbol ♈.

Instead of defining the position of a star with reference to the horizon and the meridian, that is by its azimuth and altitude, we can define its position with reference to the celestial equator, taking ♈ as the zero point of reference. The following definitions should be remembered as they are in frequent use in all works on mathematical astronomy.

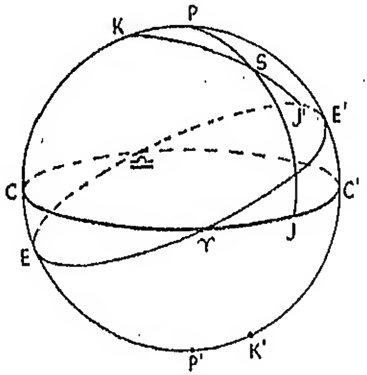

FIG. 12
The celestial sphere showing the equator, the ecliptic, and the first point of Aries, ♈.

The *declination* of a star is its distance from the equator measured by the arc of the great circle which passes through the star and the pole. The declination can be north or south according to the side of the celestial equator on which the star is situated.

The *right ascension* of a star is the arc of the equator from the first point of Aries to the foot of the perpendicular on the equator from the star. It is measured eastward from 0° to 360°.

In Fig. 12, where CC' is the equator, EE' the ecliptic, and P the north celestial pole, S is a star, and the great circle through P and S intersects the equator at J. The arc SJ is the *declination* of the star. Instead of defining the position of the star by specifying its declination it is sometimes more convenient to do so by giving its polar distance. The arc SP is known as the star's *north polar distance* and is the complement of its declination because the arc PJ is 90°, and hence SP is $90° - SJ$.

The arc ♈ J, measured from the first point of Aries to the foot of the perpendicular J, from the star to the equator, is the *right ascension*. In

the figure the point J falls to the east of ♈ but it may lie anywhere on the equator. The right ascension is usually reckoned in hours, minutes, and seconds, not in degrees, and it is easy transforming the right ascension reckoned in time into degrees, or vice versa, when this is necessary.

The celestial sphere completes a revolution in 24 hours, that is, it turns through 360° in this time, and hence in 1 hour it turns through 15°. Since there are 60 minutes of arc in a degree and 60 seconds of arc in a minute, the following relations are obvious:

$$
\begin{array}{lll}
1^h = 15^\circ & \ldots & 1^\circ = 4^m \\
1^m = 15' & \ldots & 1' = 4^s \\
1^s = 15'' & \ldots & 1'' = \frac{1^s}{15}
\end{array}
\quad \ldots \quad \ldots \quad (9)
$$

Thus, if we wish to convert the right ascension of a star, given as $3^h\ 12^m\ 30^s$, into degrees, minutes and seconds of arc, we proceed as follows:

$$
\begin{array}{lllll}
3^h & \ldots & \ldots & 45^\circ & 00' & 00'' \\
12^m & \ldots & \ldots & 3^\circ & 00 & 00 \\
30^s & \ldots & \ldots & & 7 & 30 \\
\multicolumn{3}{r}{3^h\ 12^m\ 30^s =} & 48 & 07 & 30
\end{array}
\quad \ldots \quad \ldots \quad (10)
$$

The Sidereal Day

Up to the present no definition has been given of the word "day", which has been loosely described as a period of 24 hours, an hour being 60 minutes, and a minute 60 seconds. There are two kinds of days—the ordinary day, which is determined from the motion of the sun and about which more will be said later, and the sidereal day, which is nearly 4 minutes shorter than the ordinary day. For the present we shall confine our attention to the latter.

The sidereal day is the time taken by the whole system of stars to make a complete revolution from east to west. Owing to the fact that the sun, while sharing in this revolution, has also an independent motion from west to east, the solar day differs from the sidereal day.

A sidereal clock, if set for the same instant as an ordinary clock, will soon show a discrepancy in the time, gaining about 4 minutes each day. The time for setting a sidereal clock is determined by the first point of Aries; the clock marks $0^h\ 0^m\ 0^s$ when this point crosses the meridian of the place at which observations are made and is different for different places. Hence the definition of a sidereal day is "the interval between two consecutive transits of the first point of Aries"; and the sidereal time at any instant is the number of sidereal hours, minutes and

seconds that have elapsed since the preceding transit of this point. Thus, when the sidereal time is 1^h the first point of Aries is 15° *west* of the meridian.

Hour Angle

The hour angle of a star is the angle which the star's declination circle makes with the meridian. Thus, in Fig. 13, the hour angle is $\widehat{SPZ}$ and it is measured *westwards* from the observer's meridian from 0° to 360° or from 0^h to 24^h. A scheme for converting hours, etc., into degrees

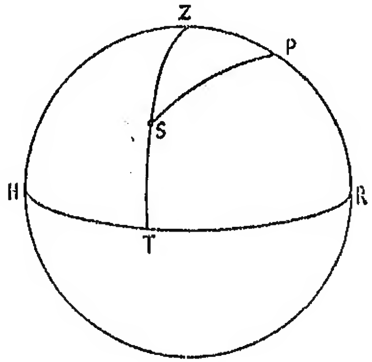

FIG. 13
Explanation of the hour angle of a celestial body.

has just been given, and it is sometimes necessary for certain purposes to make the conversion from hours to degrees. Notice in Fig. 13 that the equator is not drawn. This is unnecessary, because the great circle *PS*, drawn from the pole to the star, is the star's declination circle.

The hour angle of a star which is on the observer's meridian is 0^h, and as the heavens are moving from east to west, the star's hour angle immediately after it is on the meridian exceeds 0^h. By setting the globe for any latitude, marking the position of a star on it, and then rotating the globe, it will be found that after crossing the meridian the star will set (unless it is a circumpolar star, but it is better for the present to deal with stars that rise and set), and some time after setting it will reach the meridian again at its maximum distance *below* the horizon. The arc through which the globe has been turned from the instant when the star crossed the meridian at X' (see Fig. 14 (a)) to the instant when it reaches the meridian at X''—its maximum distance below the horizon—will be found to be 180° or 12^h. During all this time the star has been in the western hemisphere, or, in other words, its azimuth is west, and this

applies to all stars. *So long as their hour angle lies between* 0^h *and* 12^h *their azimuth is west.*

If the globe is turned after the star reaches the meridian at X'' the star passes into the eastern hemisphere and after a time it will rise at K. Its hour angle from X'' to X' where it crosses the meridian again exceeds 12^h. At X' it is 24^h or 0^h, and *during this time its azimuth is east.* Just as the star attained its maximum distance below the horizon at X'', so it attains its maximum distance above the horizon at X'.

When it is necessary to draw a diagram showing the positions of an

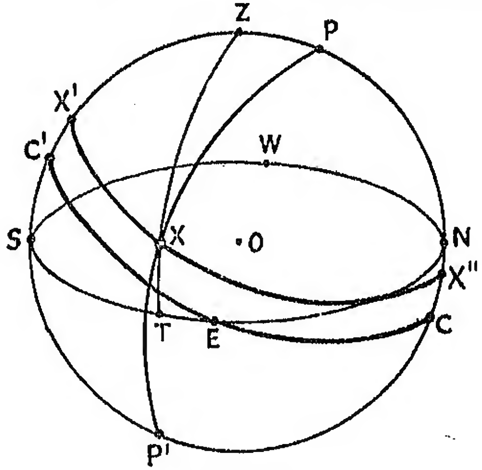

Fig. 14 (a)
The celestial sphere when the observer is in a northern latitude and observing a star in the eastern hemisphere.

observer in latitude and also of a star, etc., the following conventions should be observed.

Imagine that the observer is in northern latitude and that the star is in the eastern hemisphere. The position of the zenith Z is always taken at the top of the diagram, and having settled this point the horizon NS can be inserted, but it is necessary to decide on the positions of N and S. As the star or other heavenly body is in the eastern hemisphere, E must be placed *on the side of the horizon nearer to the reader.* The line EOW drawn through the centre O of the sphere intersects the horizon at W and the points N and S are inserted in accordance with the usual convention. The north celestial pole P must be placed so that $NP = \phi$. Fig 14 (a) shows the various positions, the star X being north of the equator CC', but the same diagram will do if the declination of the star is south.

When the star is in the western hemisphere the zenith and horizon are settled in the same way, but now the point W must be placed *on the*

side of the horizon nearer to the reader. Having fixed this point the other points on the horizon are marked according to the usual convention, that is, if the west is on the left the north is straight ahead, and so on. The north celestial pole P is placed so that $NP = \phi$, just as it is when we are dealing with the eastern hemisphere. (See Fig. 14 (b).)

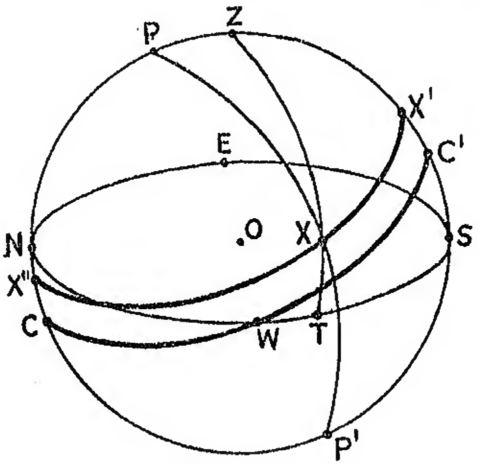

FIG. 14 (b)
The celestial sphere when the observer is in a northern latitude and observing a star in the western hemisphere.

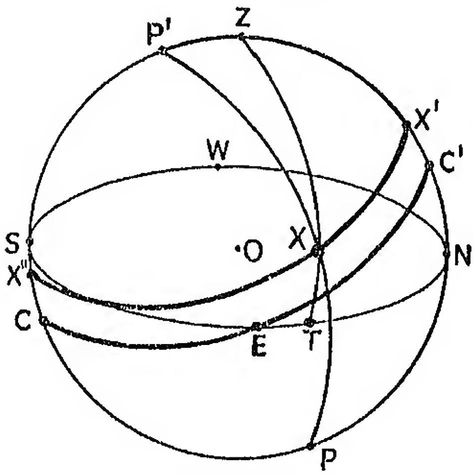

FIG. 15 (a)
The celestial sphere when the observer is in a southern latitude and observing a star in the eastern hemisphere.

If the observer is in southern latitude and the star is east, the positions are shown in Fig. 15 (a). The zenith and horizon are settled in the same way as for an observer in northern latitude, remembering that the zenith is overhead whatever be the position of the observer, and hence Z is at the top of the diagram. As the star is in the eastern hemisphere the point E on the horizon is on the side nearer to the reader and the other cardinal points are then inserted in the usual way. P' represents the south pole of the heavens and the arc $SP' = \phi$.

Fig. 15 (b) shows the diagram for a star in the western hemisphere,

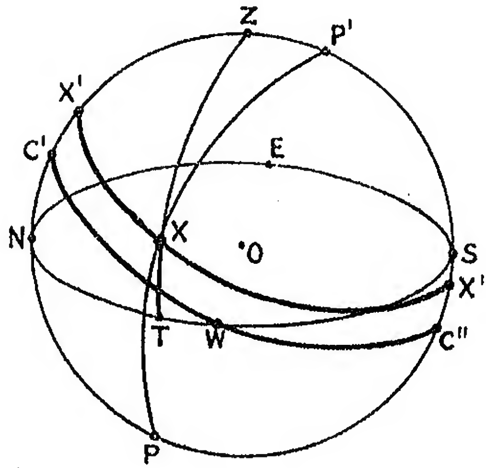

FIG. 15 (b)

The celestial sphere when the observer is in a southern latitude and observing a star in the western hemisphere.

and the positions of the cardinal points, etc., are decided in the usual way, W in this case being on the side of H nearer to the reader.

When the star is in the eastern hemisphere, as shown in Figs. 14 (a) and 15 (a), its hour angle is between 12^h and 24^h and is measured by $24 - \widehat{ZPX}$ or $24 - \widehat{ZP'X}$. When it is in the western hemisphere its hour angle is between 0 and 12^h and is measured by $\widehat{ZPX}$ or $\widehat{ZP'X}$.

As the azimuth is measured from N eastwards or westwards, in the four diagrams the azimuth is the angle PZX or $P'ZX$. It should be pointed out that a spherical angle can never exceed 180°, and hence $\widehat{ZPX}$ or $\widehat{ZP'X}$ cannot exceed 180° or 12^h.

When a star is on the meridian one half of its visible path is accom-

plished. Thus in Fig. 16, if T' and T'' are the positions of a star at rising and setting respectively, and the star is on the meridian at M, the arcs MT' and MT'' are equal.

Latitude and Longitude of a Heavenly Body

Just as the right ascension and declination of a heavenly body are referred to the equator as the fundamental plane, so the longitude and latitude are referred to the ecliptic. The *latitude* of a heavenly body is its distance from the ecliptic measured by the arc of the great circle which passes through the star and the pole of the ecliptic. In Fig. 12

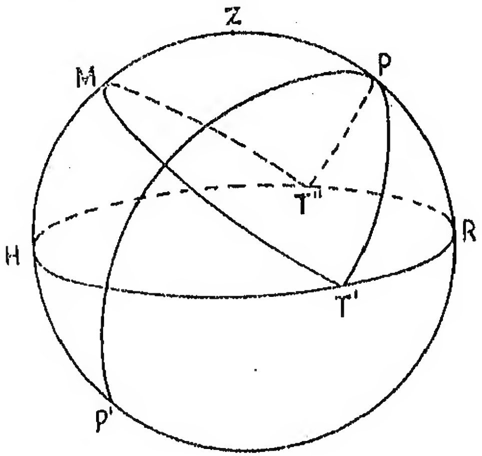

FIG. 16

Showing that a star has completed one half of its visible course in the heavens when it is on the meridian.

K is the pole of the ecliptic and KSJ' a great circle through K and the star S meeting the ecliptic in J' · SJ' is the latitude of the star.

The *longitude* of a heavenly body is the arc of the ecliptic from the first point of Aries to the foot of the perpendicular on the ecliptic from the star. It is measured eastward from 0° to 360° in degrees, minutes and seconds, and never in hour, minutes and seconds, like the right ascension. In Fig. 12 ♈ J' is the longitude of the star S. Right ascension and declination on the celestial sphere correspond to longitude and latitude on the terrestrial sphere. Longitude and latitude on the celestial sphere are not used as much as right ascension and declination, and in this work reference to these co-ordinates is necessary only on a few occasions.

The Right Ascension of a Star is the Sidereal Time of its Transit

This important relation can be very easily proved by using the globe. Suppose the R.A. of a star is 1^h or $15°$ (we are not concerned with its declination at the present) and the globe is rotated until the star is on the meridian. It will be seen that $15°$ is on the meridian and we have shown that a sidereal clock reads $0^h\ 0^m\ 0^s$ when the first point of Aries is on the meridian, and 1^h ($15°$) when the first point of Aries is $15°$ west of the meridian. Hence when the star is on the meridian the first point of Aries is $15°$ west of the meridian, and the right ascension of the star (1^h) is simply the sidereal time of its transit.

Upper and Lower Culmination of a Star

It has been shown that some stars are circumpolar, neither rising nor setting. When the hour angle of a circumpolar star is zero the star is said to be in *upper transit* or *upper culmination*, and when the hour angle is 12^h the star is said to be in *lower culmination*. It is easily seen that in the former case the star is *above* the pole, and in the latter case it is *below* the pole. The upper culmination can take place between the pole and the zenith, when it is on the north side of the zenith, or it may take place on the side of the zenith remote from the pole, when it is on the south side of the zenith.

Certain formulae are given in most text-books for dealing with problems connecting a star's declination and meridian altitude with the latitude of the place. In some cases these formulae are liable to produce some confusion if adhered to rigorously, and the reader is advised to work out each case for himself, without necessarily memorizing formulae, and to check the results, where possible, by using a globe.

Example 1

If the declination of Vega is $38° 44'$, what is its meridian altitude in latitude $51° 30'$ N?

Problems of this nature should be attacked first of all by drawing a diagram like Fig. 17. In this Z is the zenith, HR is the horizon, which it is convenient to make parallel to the top and bottom of the sheet of paper, and P is the north pole of the heavens, the arc RP being $51° 30'$. The equator can be drawn if desired, but as a number of great circles is liable to lead to confusion it will be better to do without it where it is possible. Since the declination of Vega is $38° 44'$, its north polar distance (N.P.D. is used to express this) is $90° - 38° 44' = 51° 16'$. Let V be the position of Vega so that $PV = 51° 16'$. Notice that

$PZ = 90 - \phi = 38° \ 30'$, and therefore V lies south of the zenith since $PV > PZ$.

The meridian altitude is HV, the great circle HPR being the observer's meridian, and we easily obtain the following relations:

$$HV = HP - PV$$
$$\text{But } HP = HPR - PR = 180° - \phi$$
$$\text{Hence } HV = 180° - 51° \ 30' - 51° \ 16' = 77° \ 14'.$$

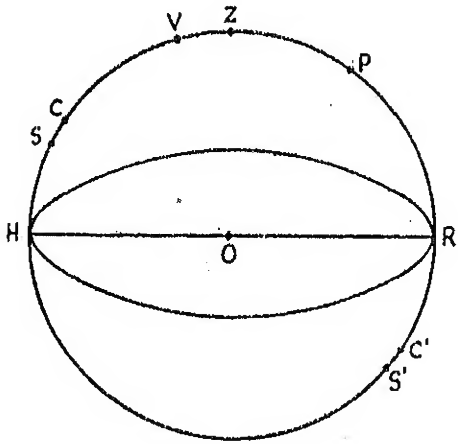

FIG. 17

Showing how the relation between the latitude of a place, the declination of a star and its meridian altitude is determined.

Instead of expressing the above in this form it is obvious that, since 180° can be written as 90° + 90°, we have

$$HV = 90° - \phi + 90° - \text{N.P.D.} = \text{colatitude} + \text{declination.}$$

The latter form is sometimes used to find a star's meridian altitude,

$$\text{meridian altitude} = \text{colatitude} + \text{declination} \quad .. \quad (11)$$

If the declination is south the negative sign is used before the declination.

It is possible to derive (11) by drawing the equator CC' in Fig. 17, the point C being between H and V.

The arc HC is equal to the arc $C'R$, and since $C'R$ is $90° - PR = 90° - \phi$, because $PC' = 90°$, the pole being always 90° from the equator,

HC is $90° - \phi$, which is the colatitude of the place. The arc CV is the declination, and therefore

$$HV = HC + CV = \text{colatitude} + \text{declination}.$$

The use of a general formula like the last one can lead to errors unless some care is exercised, for which reason a diagram is always a great advantage.

Example 2

The declination of δ Draconis is $67° 34'$. What is its altitude when it is on the meridian of Birmingham, $\phi = 52° 59'$ N.?

The N.P.D. is $22° 26'$ and if D is the position of the star on the meridian (Fig. 18), $DR = DP + PR = 22° 26' + 52° 59' = 75° 25'$. The distance RZ from the horizon to the zenith being always $90°$, the star must lie between the zenith and the pole. Hence if we took the arc HD as its altitude we should find that this exceeded $90°$, which is absurd, because the altitude of a star can never exceed $90°$. The altitude must be reckoned from R to D in this case, and is $RP + PD$ or ϕ + N.P.D. = $75° 25'$. If we had used the expression derived above

$$\text{meridian altitude} = \text{colatitude} + \text{declination}$$

the result would have been $37° 01' + 67° 34' = 104° 35'$. This would be the length of the arc HD, and to obtain the length of the arc RD, which is the star's altitude, it is necessary to deduct $104° 35'$ from $180°$, the result being $75° 25'$, the same as that previously obtained.

In this example we have dealt with the upper culmination of δ Draconis, and it remains to deal with the problem when the star is in lower culmination.

Let D' be the position of the star at lower culmination. Its altitude is RD' and from the diagram

$$RD' = RP - PD' = \phi - \text{N.P.D.}$$

The N.P.D. remains unaltered and hence

$$\text{meridian altitude} = 52° 59' - 22° 26' = 30° 33'.$$

If we add the meridian altitudes of the star at upper and lower culmination and divide by 2 the result is $52° 59'$, which is the latitude of the place. This rule always holds and is easily proved from Fig. 18, which can be taken to represent the upper and lower culminations of any star.

$RD = RP + PD$, $RD' = RP - PD'$, hence by addition
$$RD + RD' = 2RP = 2\,\phi$$

EXAMPLE 3

What must be the declination of a star which just reaches the horizon at lower culmination in the latitude of Birmingham?

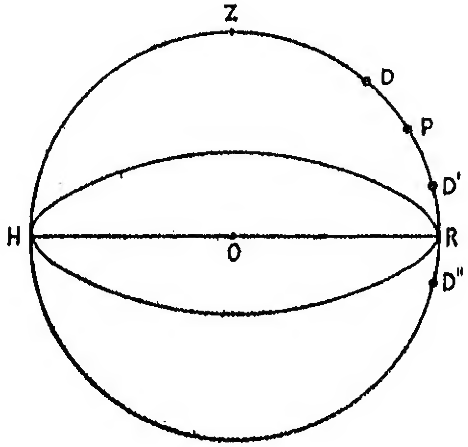

FIG. 18
Showing how the relation established by Fig. 17 requires some modification according to circumstances.

From Fig. 18, which can be used for this case also,

$$RD' = RP - PD' = \phi - \text{N.P.D.}$$

When the star is on the horizon $RD' = 0°$, hence in these circumstances,

$$\phi = \text{N.P.D.}$$

The N.P.D. of the star is, therefore, 52° 59′, and hence its declination is 90° − 52° 59′ = 37° 01′. This is the same thing as stating that the declination of the star must be the same as the colatitude of the place of observation.

If the star sets at lower culmination the point D' will be below the horizon HR. Denoting this point by D'' it is obvious that $PD'' > PR$, or N.P.D. $> \phi$. Hence, in order that a star should set at lower culmination, its N.P.D. should be greater than the latitude of the place.

Example 4

The declination of α Aquilae (Altair) is $8° 43'$. At what latitude in the northern hemisphere is it just a circumpolar star, and at what latitudes does it set?

N.P.D. $= 81° 17'$, and hence in latitude $81° 17'$ Altair just reaches the horizon. In order that the star should set at lower culmination its N.P.D., $81° 17'$, should exceed the latitude of the place. Hence at all places with latitude less than $81° 17'$ N. Altair will set. This can be verified roughly on the globe by setting it to latitude $81°$ and noticing that a star with declination $+ 8°$ just skims the horizon when its hour angle is 12^h.

Example 5

What is the meridian altitude of Altair at a place whose latitude is $40°$ S.?

When dealing with problems involving latitudes and declinations with different signs, it is always better to take the latitude $+$ whether it be the northern or southern hemisphere, and to take the declination $+$ or $-$ according to whether it is in the same or in a different hemisphere. Hence, in the present case, $\phi = 40°$, $\delta = - 8° 43'$. We can take P as the south pole of the heavens and S to be the star, which is on the side of the equator opposite to P (Fig. 17).

$\therefore$ Hence $SC = 8° 43'$.
Since $HC = RC' = 90° - \phi = 50°$,
and $HS = HC - SC = 50° - 8° 43' = 41° 17'$,

the meridian altitude is $41° 17'$.

It should be noticed that the star is on the meridian again at S', but in this case it is *below* the horizon by a distance measured by the arc RS'. In this case $RS' = RC' + C'S' = 50° + 8° 43' = 58° 43'$.

Example 6

The declination of α Canis Majoris (Sirius) is $- 16° 38'$. (a) What is its meridian altitude at a place where the latitude is $50°$ N.? (b) What is its meridian altitude at a place in latitude $50°$ S.?

(a) $90° - 50° - 16° 38' = 23° 22'$
(b) $90° - 50° + 16° 38' = 56° 38'$

Problems Involving Right Ascension

Up to the present we have dealt only with the declinations of stars, not with their right ascensions, which have not entered into the problems.

The right ascensions were irrelevant in the various stars considered, but now some problems involving right ascension, not declination, will be dealt with.

Example 7

The R.A. of α Tauri is $4^h\ 32^m\ 46^s$. A sidereal clock records the local sidereal time as $7^h\ 22^m\ 50^s$. What is the star's hour angle?

Problems of this kind are much better handled by using a globe. Even if it yields only very rough results it will show the principle involved.

On the globe mark a star with R.A. $4^h\ 32^m\ 46^s$; its declination can be anywhere, but it is more convenient to make it small so that the star is close to the equator. Rotate the globe until $7^h\ 22^m\ 56^s$ is on the meridian. The globe now represents the conditions of the celestial sphere at the moment and it will be noticed that ♈ is $7^h\ 22^m\ 56^s$ west of the meridian. Of course, accuracy to a minute cannot be obtained on the globe, but this is immaterial. Remember the definition of the sidereal time at any instant. It is the number of sidereal hours, minutes and seconds that have elapsed since the preceding transit of ♈, and obviously the conditions are fulfilled by setting the globe with $7^h\ 22^m\ 56^s$ on the meridian. The star is west of the meridian and hence its hour angle lies between 0^h and 12^h. You can place the star on the equator if you wish, because the great circle through the pole and the star will intersect the equator at $4^h\ 32^m\ 46^s$ wherever the star may be. Deducting $4^h\ 32^m\ 46^s$ from $7^h\ 22^m\ 50^s$, the arc S ♈ is found to be $2^h\ 50^m\ 04^s$, if S is the position of the star on the equator or if it is the foot of the perpendicular through the star to the equator. This arc is the same as the spherical angle SP ♈ and is the hour angle of the star. Expressing the hour angle in degrees, minutes and seconds, we proceed as follows:

2^h	..	..	30°	00′	00″
50^m	..	..	12	30	00
4^s	..	..	0	01	00
H.A.	..	..	42	31	00

From the above example we can generalize about the relation between R.A. and hour angle. This relation is

H.A. of a star = local sidereal time − star's R.A.
or H.A. of a star + star's R.A. = local sidereal time .. (12)

A general proof of (12) appears on p. 67.

EXAMPLE 8.

What is the H.A. of α Tauri if the local sidereal time is $2^h\ 10^m\ 15^s$?

In this case it is impossible to deduct the star's R.A. from the local sidereal time, so we add 24^h to the latter, and the computation is as follows:

local sidereal time	26^h	10^m	15^s
star's R.A. ..	4	32	46
H.A. of star ..	21	37	29

Since the H.A. exceeds 12^h the star's azimuth is east, a result which can be easily checked on a globe.

21^h	$315°$	$00'$	$00''$
37^m	9	15	00
29^s	0	7	15
H.A. of star ..	324	22	15

Local Sidereal Time

Local sidereal time has been referred to in all cases and this is the time that would be shown by a sidereal clock at the place with which we are dealing. Nothing has been said about the longitude of the place because this is immaterial so long as the *local sidereal time* is given. The *Nautical Almanac* supplies the sidereal time of the Meridian of Greenwich for each day of the year for Greenwich Civil Time 0^h, and the sidereal time for any other hour can be computed from this by a method which will be described later. The problem confronting us at the moment is that a sidereal clock at any place, say Greenwich, does not record the same sidereal time as another sidereal clock somewhere else, say at Petrograd, and it is necessary to have some means for converting the sidereal time at one place into that at another place.

The celestial sphere revolves through 360° in 24 sidereal hours or through 15° in one sidereal hour, and hence if a sidereal clock at Greenwich shows that the sidereal time is 10^h a sidereal clock at a place 15° east of Greenwich will indicate 11^h and at a place 15° west of Greenwich it will read 9^h. This is obvious from the fact that the transit of ♈ occurred at the place 15° east of Greenwich 1^h before it took place at Greenwich, and it occurred at the place 15° west of Greenwich 1^h after it took place at Greenwich. Hence, to obtain the sidereal time of a place east of Greenwich it is only necessary to add the longitude of the place to the Greenwich sidereal time, and to obtain the sidereal time of a place west

of Greenwich the longitude must be deducted from the Greenwich sidereal time. Longitudes east of Greenwich are reckoned — and those west of Greenwich are reckoned +, and hence the following rule can be applied in all cases, λ denoting the longitude of the place under consideration:

$$\text{local sidereal time} = \text{Greenwich sidereal time} - \lambda \quad .. \quad (13)$$

Example 9

The sidereal time at Greenwich is $4^h\ 12^m\ 16^s$. What is the sidereal time at (a) Pulkovo, $\lambda = -2^h\ 01^m\ 18^s.57$; (b) U.S. Naval Observatory, Washington, $\lambda = +5^h\ 08^m\ 15^s.75$?

(a) Sidereal time at Greenwich ..	4^h	12^m	$16^s.00$
Longitude of Pulkovo	−2	01	18·57
Sidereal time at Pulkovo	6	13	34·57
(b) Sidereal time at Greenwich ..	4	12	16·00
Longitude of Washington	+5	08	15·75
Sidereal time at Washington ..	23	04	00·25

Notice in (b) that 24^h is added on to the Greenwich sidereal time as otherwise the longitude of Washington could not be deducted from it (see also Ex. 8).

If the sidereal time at any place other than Greenwich is given the same method enables us to convert it into the sidereal time at Greenwich. In this case the formula is

$$\text{Greenwich sidereal time} = \text{local sidereal time} + \lambda \quad .. \quad (14)$$

Example 10

The longitude of Urania Observatory, Vienna, is $-1^h\ 05^m\ 33^s.48$, and the sidereal time there is $15^h\ 21^m\ 14^s.35$. What is the hour angle of α Bootis (Arcturus) at Greenwich at that time, if the R.A. of α Bootis is $14^h\ 13^m\ 07^s.54$?

Local sidereal time	15^h	21^m	$14^s.35$	
Longitude of Vienna ..	−1	05	33·48	
Greenwich sidereal time ..	14	15	40·87	
Greenwich ~~Local~~ sidereal time	14	15	40·87	
Star's R.A.	14	13	07·54	
H.A. of star	0	2	33·33	(see Ex. 7)

In the Examples which follow, the declination of a star is given to the nearest minute of arc, which is sufficiently accurate for the present purpose. Readers are strongly advised to draw diagrams and not to depend entirely on formulae; by doing so they will gain a much better knowledge of the subject than can be acquired by merely memorizing formulae.

Hints on methods of solution are given for some of the exercises on pp. 159–62.

A Celestial Globe

The upper figure in the Frontispiece shows a celestial globe the diameter of which is 21 inches. The lower figure shows a simple globe which can be made by anyone who can secure a small wooden sphere. The diameter of the sphere in the lower figure is 9 inches, but a smaller sphere than this will suffice for demonstration purposes. This sphere is capable of rotating on pivots at the poles, P being the north pole, these pivots being inserted into the circular piece of brass MM' which represents the meridian of the observer. The horizon HR can be made out of plywood or stout cardboard for a small sphere. The meridian MM' fits loosely into two slots in the horizon and rests on a small support at its lowest point, not shown in the figure. It can be moved round in its own plane so that the pole can be set at any elevation above the horizon. The meridian should be graduated and intervals of about 5°, or even 10°, are good enough for illustrative purposes. In the diagram the intervals are 10°.

The celestial equator C is shown by the thick white circle and the ecliptic E by the finer circle. It is also advisable to graduate these in intervals of about 5°. Graduation at intervals of a degree is difficult on a small sphere and it is possible to estimate approximately the intermediate positions for 5° intervals. At every third graduation on the equator, that is, at intervals of 15°, the hours should be marked, ♈ being the zero point, 15° the first hour, 30° the second hour, and so on.

From the list of positions of a few stars given on p. 63 it is possible to mark these on the globe, and this will be found useful as a check on some of the computations.

The horizon also can be graduated if the reader wishes to check the results of the computations of azimuths. Starting from the north point the scale should extend through 180° east and west. To check altitudes or zenith distances a metal strip known as the quadrant is essential. This is shown in the lower diagram and is marked Q. It should be graduated from 0° to 90°, and when the azimuth and altitude of a star are to be determined the procedure is as follows.

Suppose the latitude of the place is 50° and that the local sidereal time is 4^h, which can be computed from the Greenwich sidereal time at

0^h and the G.M.T. at which the observation is made. If the longitude is not that of Greenwich the usual corrections can be made (see p. 49). Set the globe so that the arc from P to the horizon is 50° and then rotate the globe until 4^h is on the meridian. The globe now represents the celestial sphere for latitude 50° and local sidereal time 4^h. Place the 90° graduation of the quadrant on the position marked 40° on the meridian ($90° - 50° = 40°$), and notice that the zero of the quadrant just reaches the horizon. This is a check on the accuracy of the graduations because the position marked 40° on the meridian is in the zenith, which is 90° from the horizon.

Retaining the 90° on the quadrant on the zenith, pass the quadrant through any selected star marked on the globe and take the reading which gives the altitude of the star. Take also the reading at the point on the horizon where the quadrant touches it; this is the azimuth of the star. It is more difficult to determine the azimuth accurately than it is to determine the altitude, especially when the star is near the zenith. In the latter case a small error in placing the quadrant on the star may lead to a considerable error in the reading of the azimuth, but the procedure is intended merely as a rough check on the results obtained by the accurate computations.

A Glass Model of the Celestial Sphere

If a home-made celestial globe cannot be constructed a very good rough model can be made from a spherical glass flask. It should be half filled with coloured water which will represent the horizon for all positions of the flask. Through the cork a piece of thin metal is inserted, one end projecting a few inches and the other end touching the water. The point of contact represents the observer on the horizon and the piece of metal represents the earth's axis.

Pieces of paper pasted on the outside of the flask in various positions can be used to represent the stars, and the sphere can be set for different latitudes by tilting it so that the "axis" has various inclinations to the horizon.

By rotating the glass sphere many of the phenomena referred to in the text will be obvious.

Problems

1. An observer is in latitude 38° 42′ N and observes a star in his zenith. What is the declination of the star?

2. At the equinoxes the sun's declination is zero, and at the summer and winter solstices his declination is + 23° 27′ and − 23° 27′ respectively. What is the sun's meridian altitudes on these four occasions at a place in latitude 53° N.?

3. On June 1 the sun's declination is approximately $+22°$. What is the lowest latitude at which you would just be able to see the sun all the night on this date?

4. The altitudes of a star at upper and lower culmination are observed to be $77° 18'$ and $17° 12'$ respectively. What is the latitude of the place of observation?

5. The declination of ϵ Canis Majoris is $-28° 54'$. At what latitude would it appear on the horizon at the time of its transit?

6. The declination of β Centauri is $-60° 06'$. Find its meridian altitude at a place whose latitude is $70°$ S. What is its meridian altitude if the observer is in latitude $20°$ N.?

7. If the meridian altitude of the sun is $10°$ on the shortest day of the year, what is the latitude of the place?

8. A sidereal clock at Greenwich records the sidereal time as $22^h\ 10^m\ 34^s.78$. What is the sidereal time at Sydney, New South Wales, $\lambda = -10^h\ 04^m\ 38^s$?

9. In Exercise 8 what is the hour angle of Sirius (R.A. $= 6^h\ 43^m$) at Sydney?

10. If the hour angle of a star is $2^h\ 51^m\ 02^s$ and the local sidereal time is $4^h\ 17^m\ 20^s$, find the star's right ascension.

11. Show by setting the globe that the sun rises at 6^h and sets at 18^h on March 21 and September 23 whatever the latitude of the place may be.

Chapter III

ELEMENTARY FORMULAE IN SPHERICAL TRIGONOMETRY

Up to the present the calculations have not involved any knowledge of spherical trigonometry, but a working acquaintance with this subject is necessary before proceeding to certain computations which are in constant use in astronomy. A few words follow on spherical triangles and on some of the formulae frequently required.

A spherical triangle is the figure on the surface of a sphere bounded by three arcs of *great circles*. A small circle cannot form the side of a spherical triangle, and when it becomes necessary to deal with small circles the method of treatment differs completely from that employed in the case of great circles. See (1) for a case of a small circle and the relation between its arc and that of a great circle.

If O is the centre of a sphere (Fig. 19) and OAB, OAC, OBC are three planes through O intersecting the surface of the sphere in the arcs AB, AC and BC respectively, ABC is a spherical triangle. The angles of this spherical triangle are the inclinations of the three planes; thus the angle A is the inclination of the planes OAC and OAB; the angle B is the inclination of the planes OBC and OBA; and the angle C is the inclination of the planes OCB and OCA. The sides of the spherical triangle are arcs of great circles and hence in dealing with spherical triangles we are concerned primarily with angles and arcs, not with lengths as in the case of plane triangles. Of course the lengths of the arcs can be determined when the radius of the sphere is known.

The following elementary formulae are important and proofs will be found in any treatise on spherical trigonometry. Other formulae will be given as required, but those numbered (a), (b) and (c) are all that are necessary as a basis for the present chapter.

Let ABC be a spherical triangle, A, B and C denoting the angles at A, B and C, and a, b, c denoting the sides opposite each of these angles.

$$\begin{aligned}
&\text{(a)}\quad \cos a = \cos b \cos c + \sin b \sin c \cos A \\
&\text{(b)}\quad \sin A/\sin a = \sin B/\sin b = \sin C/\sin c \\
&\text{(c)}\quad \cos A = (\cos a - \cos b \cos c)/\sin b \sin c \qquad \ldots \quad (15)
\end{aligned}$$

This last formula is derived from (a) and so is not an independent formula.

It is possible to write (a) in various forms according to the side that

we wish to find. Thus, if we require the sides b and c, (a) can be expressed as follows:

$$\cos b = \cos c \cos a + \sin c \sin a \cos B$$
$$\cos c = \cos a \cos b + \sin a \sin b \cos C \quad \ldots \quad \ldots \quad (15\text{A})$$

It will be seen that 15 (a) requires that two sides and the included angle be given, from which it is possible to calculate the third side, while (b) requires that two angles and an opposite side or two sides and an opposite angle be given, from which another opposite side or another opposite angle can be found.

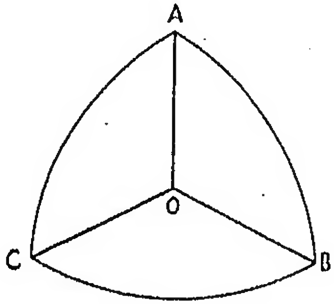

FIG. 19
A spherical triangle.

Transformation of Co-ordinates

The first problem that will be solved by the aid of these formulae is connected with the transformation from the equatorial system of co-ordinates—R.A. and dec.—to the horizontal system—altitude and azimuth. Instead of the dec. the N.P.D. or S.P.D. will sometimes be used, and the zenith distance will frequently take the place of the altitude.

Problem 1. Given the latitude ϕ, the R.A. α, the declination δ, of a star, and the local sidereal time θ, find its azimuth A and zenith distance z.

In Fig. 20 let P be the pole, CC' the equator, Z the zenith, HR the horizon, HZR the meridian, and S a star which we will suppose is on the west side of the meridian. The arc PS meets the equator in J and hence the star's right ascension is the arc ΥJ. The hour angle ZPS is the difference between the local sidereal time θ and the R.A. of the star, so that $h = \theta - \alpha$.

In the triangle ZPS we have as follows:

$$SP = \text{N.P.D.} = 90^\circ - \delta$$
$$ZP = 90^\circ - \text{lat.} = 90^\circ - \phi$$
$$\text{Angle } ZPS = \text{hour angle of the star} = h$$

By 15 (a) we have

$$\cos ZS = \cos ZP \cos SP + \sin ZP \sin SP \cos h$$

But

$$\cos ZP = \cos (90^\circ - \phi) = \sin \phi, \quad \cos SP = \cos (90^\circ - \delta) = \sin \delta$$
$$\sin ZP = \sin (90 - \phi) = \cos \phi, \quad \sin SP = \sin (90^\circ - \delta) = \cos \delta$$

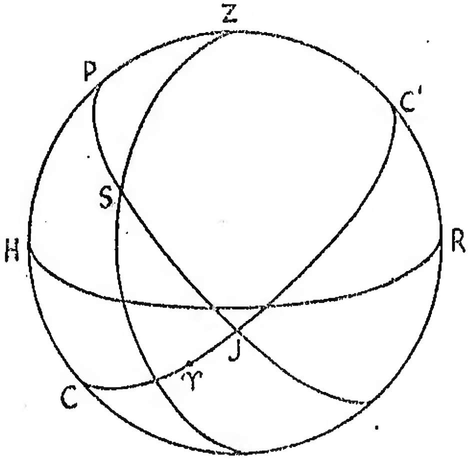

FIG. 20

Used to find the formulae for the transformation of a star's right ascension and declination into its azimuth and altitude.

Denoting ZS by z, the above reduces to

$$\cos z = \sin \phi \sin \delta + \cos \phi \cos \delta \cos h \quad \ldots \quad (16)$$

By 15 (b) we have

$$\sin \widehat{PZS}/\sin PS = \sin \widehat{ZPS}/\sin ZS$$

But $\widehat{PZS}$ measures the azimuth A of the star, and hence the above reduces to

$$\sin A/\sin h = \cos \delta/\sin z, \text{ or}$$
$$\sin A = \sin h \cos \delta/\sin z \quad \ldots \quad \ldots \quad \ldots \quad (17)$$

While (16) and (17) determine the values of $\cos z$ and $\sin A$, (17) does not determine the azimuth A uniquely because the angle A might lie in the first or second quadrant, and it is necessary to deal with this ambiguity.

From 15 (c)

$$\cos \widehat{PZS} = (\cos PS - \cos PZ \cos ZS)/\sin PZ \sin ZS$$

Hence

$$\cos A = (\sin \delta - \sin \phi \cos z)/\sin z \cos \phi \qquad \ldots \qquad \ldots \qquad (18)$$

From the manner of reckoning A, i.e. from the north eastward up to 180°, or westward up to 180°, it is obvious that A must lie in the first or second quadrant in all cases. To be consistent it will be better to make h also lie in the first or second quadrant, and this can be done as follows:

When h exceeds 12^h deduct it from 24^h and use the formulae just given, noticing that in this case the azimuth must be east. When h is less than 12^h the azimuth is west. It must be remembered, however, that the azimuth is not necessarily in the same quadrant as h, and hence $\cos A$ must be computed to decide on the quadrant. It will be better in most cases to compute $\sin A$ as well to check the results. There is no ambiguity about z, which can always be determined from $\cos z$, and z never exceeds 180°.

The use of the formulae will be illustrated by a few examples.

EXAMPLE 1

Let $\phi = 51° 30'$ N. and let the co-ordinates of a star be $\alpha = 21^h 40^m$, $\delta = +25° 12'$, and let θ, the local sidereal time, be $1^h 52^m$. Find z and A, the star's zenith distance and azimuth.

$h = 25^h 52^m - 21^h 40^m = 4^h 12^m$ (add 24^h to make the deduction of α ~~θ~~ possible). $4^h 12^m = 63°$.

Referring to (16), let $X = \sin \phi \sin \delta$, and $Y = \cos \phi \cos \delta \cos h$. Hence

$$\cos z = X + Y$$

$\log \sin \phi$	$\bar{1}\cdot 8935$	$\log \cos \phi$	$\bar{1}\cdot 7941$
$\log \sin \delta$	$\bar{1}\cdot 6292$	$\log \cos \delta$	$\bar{1}\cdot 9566$
		$\log \cos h$	$\bar{1}\cdot 6570$
$\log X$	$\bar{1}\cdot 5227$	$\log Y$	$\bar{1}\cdot 4077$
X	$0\cdot 3332$	Y	$0\cdot 2557$

$$\cos z = X + Y = 0\cdot 5889$$
$$z = 53° 55'$$

Using (17)

$\log \cos \delta$	$\bar{1}\cdot 9566$
$\log \sin h$	$\bar{1}\cdot 9499$
$\log \cos \delta \sin h$	$\bar{1}\cdot 9065$
$\log \sin z$	$\bar{1}\cdot 9075$
$\log \sin A$	$\bar{1}\cdot 9990$
A	$86° 06'$ or $180° - 86° 06' = 93° 54'$

From (18) $\cos A = (P - Q)/R$, where
$$P = \sin\delta,\ Q = \sin\phi\cos z,\ R = \sin z\cos\phi$$

$P = 0{\cdot}4258$,	$\log\sin\phi$	$\bar{1}{\cdot}8935$
	$\log\cos z$	$\bar{1}{\cdot}7701$
	$\log Q$	$\bar{1}{\cdot}6636$
	Q	$0{\cdot}4609$
	$P - Q$	$-0{\cdot}0351$
	$\log\sin z$	$\bar{1}{\cdot}9075$
	$\log\cos\phi$	$\bar{1}{\cdot}7941$
	$\log R$	$\bar{1}{\cdot}7016$
	$\log(P - Q)$	$\bar{2}_n{\cdot}5453$
	$\log R$	$\bar{1}{\cdot}7016$
	$\log\cos A$	$\bar{2}_n{\cdot}8437$
	A	$180° - 86° = 94°$

This example shows that if an angle is large—close to 90°—greater accuracy is obtained by computing its value from its cosine. When an angle is small its value should be computed from its sine. The reason is that the sine of an angle changes slowly when it is close to 90° and its cosine changes slowly when it is near 0°. In the present case an error of 6′ occurs from using the sine to determine the value of the angle. While it was unnecessary to compute $\sin A$, yet it is advisable to do so as a check on the work.

It will be noticed that $\cos A$ is $-$ and the subscript n denotes this. Hence as $\cos A$ is negative A is in the second quadrant. The azimuth is, therefore, 94° W.

Example 2

With the same data except the sidereal time, find A and z if θ is $5^{h}\ 28^{m}$.

$$h = 29^{h}\ 28^{m} - 21^{h}\ 40^{m} = 7^{h}\ 48^{m} = 117°.$$

Since $117° = 180° - 63°$, the computation is the same as that just given except that $\cos h$ is $-$. Hence $\cos\phi\cos\delta\cos h$ is $-0{\cdot}2557$, and $X + Y = 0{\cdot}0775$, so that $z = 85°\ 33'$. The remainder of the computation is as follows:

$\log\cos\delta$	$\bar{1}{\cdot}9566$
$\log\sin h$	$\bar{1}{\cdot}9499$
$\log\cos\delta\sin h$	$\bar{1}{\cdot}9065$
$\log\sin z$	$\bar{1}{\cdot}9987$
$\log\sin A$	$\bar{1}{\cdot}9078$
A	$53°\ 58'$ or $180° - 53°\ 58'$

$\log \sin \phi$	$\bar{1}\cdot 8935$
$\log \cos z$	$\bar{2}\cdot 8893$
$\log Q$	$\bar{2}\cdot 7833$
Q	$0\cdot 0607$
$P - Q$	$0\cdot 3651$
$\log \sin z$	$\bar{1}\cdot 9987$
$\log \cos \phi$	$\bar{1}\cdot 7941$
$\log R$	$\bar{1}\cdot 7928$
$\log (P - Q)$	$\bar{1}\cdot 5624$
$\log R$	$\bar{1}\cdot 7928$
$\log \cos A$	$\bar{1}\cdot 7696$
A	$53^\circ\, 58'$

Since $\cos A$ is $+$ A must be in the first quadrant. Because h is less than 12^h the azimuth is W and hence is $53^\circ\, 58'$ W.

Example 3

Now let θ be $13^h\, 52^m$ so that $h = 13^h\, 52^m - 21^h\, 40^m = 16^h\, 12^m$ Since h is greater than 12 we deduct $16^h\, 12^m$ from 24^h and obtain $7^h\, 48^m$ This case is then dealt with in the same way as the last example, and $z = 85^\circ\, 33'$, $A = 53^\circ\, 58'$ E. The azimuth is east because h exceeds 12^h.

Example 4

In the final example we shall assume $\theta = 17^h\, 28^m$ so that $h = 19^h\, 48^m$. Deducting this from 24^h we obtain $4^h\, 12^m$ and the case is similar to the first example. The zenith distance of the star is, therefore, $53^\circ\, 55'$, and its azimuth is 94° E.

All cases of transformation of a star's equatorial co-ordinates to horizontal co-ordinates can be dealt with in the same way as in the above four examples.

If we are given the latitude, azimuth and zenith distance, we can find the hour angle and the declination. The method of computation is easily seen from Fig. 20.

Using (15a),

$$\cos PS = \cos PZ \cos ZS + \sin PZ \sin ZS \cos \widehat{PZS}$$

from which

$$\sin \delta = \sin \phi \cos z + \cos \phi \sin z \cos A \qquad \text{(19)}$$

To find h we can use (15c)

$$\cos \widehat{SPZ} = (\cos ZS - \cos PZ \cos PS)/\sin PZ \sin PS$$

or

$$\cos h = (\cos z - \sin \phi \sin \delta)/\cos \phi \cos \delta \qquad .. \quad (20)$$

These formulae will be used to check the results just obtained.

Example 5

Let $\phi = 51° 30'$ N., $A = 94°$ W., $z = 53° 55'$.

$\log \sin \phi$	$\bar{1}{\cdot}8935$	$\log \cos \phi$	$\bar{1}{\cdot}7941$
$\log \cos z$	$\bar{1}{\cdot}7701$	$\log \sin z$	$\bar{1}{\cdot}9075$
$\log \sin \phi \cos z$	$\bar{1}{\cdot}6636$	$\log \cos A$	$\bar{2}_n{\cdot}8436$
$\sin \phi \cos z$	$0{\cdot}4609$	$\log \cos \phi \sin z \cos A$	$\bar{2}_n{\cdot}5452$
		$\cos \phi \sin z \cos A$	$-0{\cdot}0351$

$$\sin \delta = 0{\cdot}4609 - 0{\cdot}0351 = 0{\cdot}4258$$
$$\delta = 25° \; 12'$$

To find h we have

$\cos z = 0{\cdot}5890$	$\log \sin \phi$	$\bar{1}{\cdot}8935$
	$\log \sin \delta$	$\bar{1}{\cdot}6292$
	$\log \sin \phi \sin \delta$	$\bar{1}{\cdot}5227$
	$\sin \phi \sin \delta$	$0{\cdot}3332$

$$\cos z - \sin \phi \sin \delta = 0{\cdot}2558$$

$\log \cos \phi$	$\bar{1}{\cdot}7941$	$\log 0{\cdot}2558$	$\bar{1}{\cdot}4079$
$\log \cos \delta$	$\bar{1}{\cdot}9566$	$\log \cos \phi \cos \delta$	$\bar{1}{\cdot}7507$
$\log \cos \phi \cos \delta$	$\bar{1}{\cdot}7507$	$\log \cos h$	$\bar{1}{\cdot}6572$
		h	$62° \; 59'$

This is 1′ out, the value previously adopted being 63°, but it is easy to lose a unit or two in the fourth place in such computations. As the star is west the hour angle is taken to be 63°.

Example 6

Suppose that z is 85° 33′ and A is 53° 58′ E., find δ and h.

$\log \sin \phi$	$\bar{1}{\cdot}8935$	$\log \cos \phi$	$\bar{1}{\cdot}7941$
$\log \cos z$	$\bar{2}{\cdot}8897$	$\log \sin z$	$\bar{1}{\cdot}9987$
$\log \sin \phi \cos z$	$\bar{2}{\cdot}7832$	$\log \cos A$	$\bar{1}{\cdot}7696$
$\sin \phi \cos z$	$0{\cdot}0607$	$\log \cos \phi \sin z \cos A$	$\bar{1}{\cdot}5624$
		$\cos \phi \sin z \cos A$	$0{\cdot}3651$

$$\sin \delta = 0{\cdot}3651 - 0{\cdot}0607 = 0{\cdot}4258$$
$$\delta = 25° \; 12'$$

$\log \sin \phi$	$\bar{1}\cdot 8935$	$\cos z$	$0\cdot 0776$
$\log \sin \delta$	$\bar{1}\cdot 6292$	$\sin \phi \sin \delta$	$0\cdot 3332$
$\log \sin \phi \sin \delta$	$\bar{1}\cdot 5227$	$\cos z - \sin \phi \sin \delta$	$-0\cdot 2556$
$\sin \phi \sin \delta$	$0\cdot 3332$		

$\log \cos \phi$	$\bar{1}\cdot 7841$	$\log (\cos z - \sin \phi \sin \delta)$	$\bar{1}_n\cdot 4075$
$\log \cos \delta$	$\bar{1}\cdot 9566$	$\log \cos \phi \cos \delta$	$\bar{1}\cdot 7507$
$\log \cos \phi \cos \delta$	$\bar{1}\cdot 7507$	$\log \cos h$	$\bar{1}_n\cdot 6568$

Since $\log \cos h$ is $-$ it follows that h can be $180° \pm 62° 59'$ (a discrepancy of $1'$ occurs in comparison with the earlier computation), but as the azimuth is east h must be greater than 12^h or $180°$ and hence $h = 243° = 16^h 12^m$.

The R.A. of the star can be found when the local sidereal time is known. Thus, suppose in the last case that the local sidereal time is $13^h 52^m$, then from the expression

$$\text{H.A. of a star} = \text{local sidereal time} - \text{star's R.A.}$$

we have

$$16^h 12^m = 13^h 52^m - \text{star's R.A., or}$$
$$\text{star's R.A.} = 13^h 52^m - 16^h 12^m = 21^h 40^m.$$

Some of the problems previously considered for the particular case when a star is on the meridian can be solved by (16). Thus, suppose we want to find the conditions that a star should be on the horizon at lower culmination, it is only necessary to make $h = 12^h$ or $180°$ in (16), and $z = 90°$. Since $\cos 180° = -1$ and $\cos 90° = 0$, (16) yields

$$\sin \phi \sin \delta - \cos \phi \cos \delta = 0, \text{ or}$$
$$\cos (\phi + \delta) = 0.$$

Hence

$$\phi + \delta = 90°.$$

Since $\delta = 90° - $ N.P.D., it follows that

$\phi =$ N.P.D., a result previously obtained.

Calculation of the Times of Rising and Setting of a Heavenly Body

An important application of the formulae just derived is to determine the times of rising and setting of a heavenly body. This admits of a simple solution, since $z = 90°$ when a body is on the horizon, and hence (16) becomes

$$\sin\phi \sin\delta + \cos\phi \cos\delta \cos h = 0, \text{ from which}$$
$$\cos h = -\tan\phi \tan\delta \qquad \ldots \qquad \ldots \qquad (21)$$

The use of this formula will be illustrated by a few examples.

Example 7

The declination of the sun is $+ 18°$ about May 12 and August 21, and the latitude of the place is $50°$ N. Find the hour angle of the sun at rising and setting.

log tan ϕ	$0{\cdot}0762$
log tan δ	$\bar{1}{\cdot}5118$
log cos h	$\bar{1}_n{\cdot}5880$

$$h = 180° \pm 67° \ 13' = 12^h \pm 4^h \ 28^m \ 52^s.$$

Both values satisfy the negative result for cos h, and hence as the sun rises in the east and sets in the west, the hour angle in the former case exceeds 12^h and in the latter case is less than 12^h. Hence the hour angle at rising is $16^h \ 28^m \ 52^s$, and at setting it is $7^h \ 31^m \ 08^s$.

Example 8

If the declination of the sun is $- 18°$, find the hour angle of the rising and setting of the sun at a place in latitude $50°$ N.

The computation is the same, but since tan δ is $-$ in this case and the $-$ sign appears before the terms on the right-hand side of (21), cos h is $+$. Hence $h = 67° \ 13'$ or $360° - 67° \ 13'$, either value of h giving a positive result for cos h. In this case, therefore, the hour angle of rising is $24^h - 4^h \ 28^m \ 52^s = 19^h \ 31^m \ 08^s$, and the hour angle of setting is $4^h \ 28^m \ 52^s$.

Azimuth of a Heavenly Body at Times of Rising and Setting

The azimuth of a body at rising or setting is easily found by making $z = 90°$ in (18), which then becomes

$$\cos A = \sin\delta / \cos\phi \qquad \ldots \qquad \ldots \qquad \ldots \qquad (22)$$

Example 9

Find the azimuth of the sun at rising and setting on June 21 and December 23 when his declination is $+ 23° \ 27'$ and $- 23° \ 27'$, taking the latitude as $51° \ 30'$ N.

log sin δ	$\bar{1}{\cdot}5999$
log cos ϕ	$\bar{1}{\cdot}7941$
log cos A	$\bar{1}{\cdot}8058$
A	$50° \ 15'$

The azimuth is 50° 15′ E. at the time of rising and 50° 15′ W, at the time of setting.

On December 23, when the declination of the sun is $-23° 27'$, sin δ is $-$ and hence log cos A is $-$. In this case $A = 180° - 50° 15' = 130° 45'$. Hence the azimuth at sunrise is 130° 45′ E. and at sunset it is 130° 45′ W.

The Distance Between Any Two Points on the Earth's Surface

The distance between any two points on a great circle was found previously, in the restricted case where the points were in the same latitude. It is possible to use (15 (a)) to find the distance between two points on a great circle connecting them, their latitude being the same. The following example will show the method for computing the length of the arc of a great circle drawn through any two places on the earth's surface.

Example 10

A place is in latitude 50° N. and longitude 60° E., and another place is in latitude 16° N. and longitude 36° W. Find the great circle distance between the two places.

Mark the positions A and B of the places on a sphere and join each place to the pole P by great circles. The angle APB is the difference in the longitudes of A and B and is $60° + 36° = 96°$; $PA = 90° - 50° = 40°$; $PB = 90° - 16° = 74°$.

$$\cos AB = \cos PA \cos PB + \sin PA \sin PB \cos APB = X + Y$$

Hence

$$\cos AB = \cos 40° \cos 74° + \sin 40° \sin 74° \cos 96° = X + Y$$

log cos 40°	$\bar{1}\cdot 8843$	log sin 40°	$\bar{1}\cdot 8081$
log cos 74°	$\bar{1}\cdot 4403$	log sin 74°	$\bar{1}\cdot 9828$
log X	$\bar{1}\cdot 3246$	log cos 96°	$\bar{1}_n\cdot 0192$
X	$0\cdot 2116$	log Y	$\bar{2}_n\cdot 8101$
		Y	$-0\cdot 0646$

$$\cos AB = 0\cdot 2116 - 0\cdot 0646 = 0\cdot 1470$$
$$AB = 81° 33'$$

Example 11

Find the distance between two places A and B, the latitude and longitude of A being 60° N. and 15° E., those of B being 20° S. and 75° E.

Join P the north pole to B by a great circle. Since the arc from P to the equator is 90° the arc PB is $90° + 20° = 110°$. The angle APB is

$75^\circ - 15^\circ = 60^\circ$. The sides of the spherical triangle APB are 30° and 110° and the included angle is 60°.

log cos 30°	$\bar{1}\cdot9375$	log sin 30°	$\bar{1}\cdot6990$
log cos 110°	$\bar{1}_n\cdot5341$	log sin 110°	$\bar{1}\cdot9730$
log X	$\bar{1}_n\cdot4716$	log cos 60°	$\bar{1}\cdot6990$
X	$-0\cdot2962$	log Y	$\bar{1}\cdot3710$
		Y	$0\cdot2350$

$$\cos AB = -0\cdot2962 + 0\cdot2350 = -0\cdot0613$$
$$AB = 93^\circ\ 31'$$

The R.A. and dec. of a few stars are given below for reference. Some of the problems require these co-ordinates, which are correct to the nearest minute of arc.

Star	*R.A.*			*Dec.*	
α Geminorum	7^h	31^m	06^s	$+32^\circ$	$01'$
α Leonis	10	05	26	+12	14
ϵ Ursae Majoris	12	51	36	+56	15
α Virginis	13	22	16	−10	52
α Lyrae	18	35	02	+38	44
α Pavonis	20	21	15	−56	55
α Tucanae	22	14	42	−60	32

Problems

1. What is the local sidereal time when α Geminorum is on the meridian?

2. Find the hour angle of α Leonis if the local sidereal time is 18^h.

3. What is the hour angle at rising and setting of α Virginis in latitude 50° N.? What is the azimuth of the star in each case?

4. Find the azimuth and altitude of α Pavonis when the local sidereal time is $7^h\ 12^m\ 15^s$ at a place in latitude 40° S.

5. What is the azimuth of the sun when rising on November 1 at a place in latitude 20° S.? The sun's declination on November 1 can be taken as $-14^\circ\ 15'$.

6. The azimuth of ϵ Ursae Majoris when it is rising is 30° E. What is the latitude of the place?

7. If the sun sets at 15^h (3 p.m.) on the shortest day of the year, what is the latitude of the place?

8. At what latitudes would α Lyrae be circumpolar?

9. What is the hour angle of α Tucanae at rising and setting at a place in latitude 20° S.? What is its azimuth in each case?

10. What are the values of h and A in (9) if the latitude is 20° N.?

11. The latitude and longitude of New York are $40^\circ\ 43'$ N. and 74° W., and of Cape Town $33^\circ\ 56'$ S. and $18^\circ\ 28'$ E. What is the arc of the great circle between them and what is its length in nautical miles?

Chapter IV

PROBLEMS ARISING FROM THE SUN'S MOTION AMONGST THE STARS

The earth moves round the sun, completing a revolution in a year, but the motion is not uniform, and this fact is responsible for certain problems in the determination of time. The reason for the non-uniform motion of the earth round the sun is that the curve it describes is not a circle but an ellipse, the sun being in one focus of the ellipse. Fig. 21 shows an ellipse which can be easily traced out on a piece of paper by inserting two pins into the paper, passing a loop over them with a string, and then moving a pencil round the paper, its point keeping the string tight. It is not necessary to deal with the properties of an ellipse at this stage and a few facts relating to the motions of the planets, including the earth, will be considered.

Each pin point is in the focus of the ellipse described by the end of the pencil, and it is easily seen that the distances of different points on the ellipse from a focus vary. The same remark applies to the planets, all of which move in ellipses round the sun which is in one focus. When a planet attains its closest approach to the sun (it is then said to be at *perihelion*) its velocity in its orbit is a maximum, and when it attains its greatest distance from the sun (when it is said to be at *aphelion*) its velocity is a minimum. The distance of the earth from the sun at perihelion, about January 2 each year, is 91,449,000 miles, and its distance at aphelion on July 4 is 94,561,000 miles. Hence the earth has a greater orbital velocity on January 2 than it has on July 4, its velocity gradually decreasing from perihelion to aphelion.

The orbital motion of the earth round the sun can be represented by the motion of the sun round the earth, the earth now occupying one of the foci. This conception is in accordance with the previous hypothesis that the earth is fixed and that the whole celestial sphere is revolving round it from east to west. Hence the motion of the sun is not uniform, and as the sun is used for measuring time in the ordinary affairs of life, it is necessary to make certain assumptions about his motion if an accurate record of time is to be kept.

We have seen that the sun apparently moves in the ecliptic, this apparent motion being actually due to the movement of the earth round the sun (not to the earth's diurnal rotation). The apparent orbit of the

sun relative to the earth lies in a plane which is called the plane of the ecliptic. The earth's axis is inclined at an angle of about 66° 33′ to this plane, so that the planes of the equator and the ecliptic are inclined at an angle of 23° 27′.

It is very easy to notice that the moon has an easterly motion amongst the stars but it is more difficult to see this in the case of the sun. In climates which afford the best opportunities for observing the heavenly bodies—that of Egypt, for instance—the easterly movement of the sun amongst the stars is not difficult to detect. If the sun is observed to rise about half an hour later than a star, a few mornings afterwards it will be observed to rise more than half an hour after the same star,

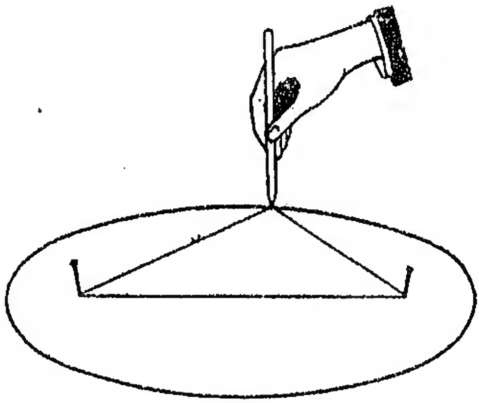

FIG. 21
Method for drawing an ellipse.

this phenomenon being due to the easterly motion of the sun with reference to the star. The same thing can be seen in many countries, but in the British Isles the weather conditions do not always permit of such accurate observations with the naked eye as they do in some other countries.

If the sun's motion among the stars were uniform and if he moved in the equator instead of in the ecliptic, less complications regarding time would arise. It may be said, however, that if the sun moved in the equator this would imply that the earth's axis would be perpendicular to the ecliptic, and we should not enjoy the changes of the seasons. Even the lengths of the days and nights would not differ, day and night being each 12 hours at every place on the earth. Probably most people would prefer the existing arrangement in spite of the fact that the time indicated by a sundial—known as *dial time*—can differ from the mean time as shown by a clock by more than quarter of an hour.

Equation of Time

In Fig. 22 E is the earth and S the sun on January 2, the sun being then at perigee* if we suppose that the earth is fixed and that the sun is moving in the direction shown by the arrow. The line ES traces out 360° in a year but not at a uniform rate, and a fictitious point known as the *dynamical mean sun* is supposed to move round E in the ecliptic at a uniform speed, completing a revolution in a year. The dynamical mean sun at perigee is in the direction ES, and since the real sun moves more rapidly near perigee than it does elsewhere it will be in advance of the dynamical mean sun at this portion of its orbit as shown, ES' being the vector from the earth to the real sun and ED the vector to the dynamical mean sun. At some other parts of its orbit the real sun will be behind

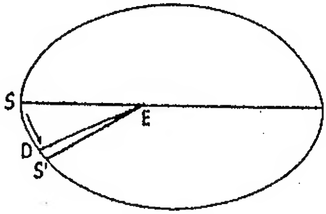

FIG. 22

The apparent motion of the sun around the earth in a year is an ellipse.

the dynamical mean sun and at apogee* its direction will coincide with it again as it does at perigee.

In Fig. 23 let G be the earth in the centre of the celestial sphere and let D be the dynamical mean sun at a point D on the *ecliptic* EE'. In addition to this fictitious point another fictitious point M called the *mean sun* moves round on the *equator* with the same angular velocity as D. These two fictitious points do not coincide at perigee but at ♈, and hence ♈ $D =$ ♈ M. The mean sun describes the circuit of the equator with reference to the stars in the same time in which the dynamical mean sun describes the circuit of the ecliptic. Since the longitude of the dynamical mean sun increases uniformly, the R.A. of the astronomical mean sun also increases uniformly, so the motion of this point gives a uniform measure of time.

A great circle from the pole P of the celestial sphere through S, the sun, meets the equator at A, and from the definition of right ascension,

* The word *perigee* is derived from the Greek *peri* near, and *ge* the earth. *Apogee* is derived from the Greek *apo* from, and *ge* the earth. The words mean "nearest to the earth" and "at greatest distance from the earth" respectively. *Perihelion* and *aphelion* are derived in the same manner, *helios* the sun being substituted for *ge* the earth.

the R.A. of S is ΥA. The R.A. of the mean sun is ΥM and the small arc AM, which is the difference of the right ascensions, is known as the *equation of time*. If the R.A. of the mean sun be denoted by R.A.M.S. and the R.A. of the true sun, or simply the sun, be denoted by R.A.$\odot$, then, E.T. denoting the equation of time,

$$\text{E.T.} = \text{R.A.M.S.} - \text{R.A.}\odot \qquad \ldots \qquad \ldots \qquad (23)$$

The hour angle of S is the angle SPC' and is measured by the arc AC'. The R.A. of S is ΥA and since $\Upsilon A + AC' = \Upsilon C'$ = local sidereal time, it follows that the hour angle of the sun + sun's R.A. = local

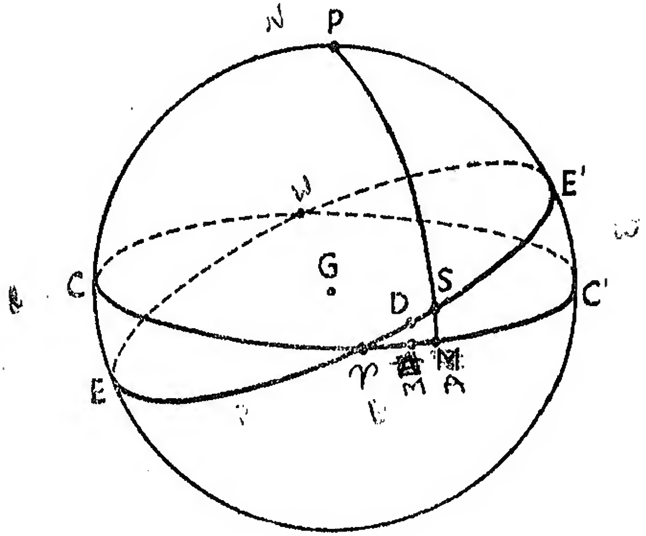

FIG. 23
Used for deriving the formula for the equation of time.

sidereal time. The same thing obviously applies to the mean sun M, and hence we obtain the relation,

H.A. sun + R.A. sun = H.A. mean sun + R.A. mean sun = local sidereal time.

This was shown to be true for a star (see (12)).

From the above relation we easily deduce

R.A. mean sun − R.A. sun = hour angle sun − hour angle mean sun.

Hence

E.T. = hour angle sun − hour angle mean sun = dial time − clock time .. .. (24)

= L.A.T — L.M.T

The value of the equation of time varies and vanishes four times in a year, on or about April 16, June 14, September 1 and December 25. Its maximum value takes place on November 3 when it is $+ 16^m 22^s$. The *Nautical Almanac* supplies the value of the E.T. for every day in the year.

Mean Solar Day

A *mean solar day* is the interval between two consecutive transits of the mean sun over the same meridian. It is divided into 24 hours of mean solar time and the hour is divided into minutes and seconds. During a year while the sun moves round a complete circuit the first point of Aries makes one more revolution about the earth than the sun does and hence we have the following relation:

$$366\tfrac{1}{4} \text{ sidereal days} = 365\tfrac{1}{4} \text{ mean solar days.}$$

From this relation the following figures are obtained:

24^h M.S.T.	..	24^h	03^m	$56^s.5560$	sidereal time			
1^h ,,	..	1	00	09·8565	,,			
1^m ,,	..	0	01	00·1643	,,			
1^s ,,	..	0	00	01·0027	,,		..	(25)

Tables are given in various works on astronomy for converting intervals of one time into intervals of the other time, and those who have a *Nautical Almanac* will find the necessary tables there. These save much labour in computation.

To convert intervals of sidereal time into mean solar time, the figures below can be used. If computations are made, tables not being available, it will be found simpler to leave the figures in the form given, as an example will show.

24^h sidereal time	..	$24^h - 3^m$	$55^s.9104$	mean solar time			
1^h ,,	..	$1^h - 0$	09·8296	,,			
1^m ,,	..	$1^m - 0$	0·1638	,,			
1^s ,,	..	$1^s - 0$	0·0027	,,		..	(26)

Suppose we wanted to convert $3^h\ 55^m\ 10^s$ sidereal time into mean solar time, we proceed thus:

$3 \times 9^s.8296$	..	..	$29^s.4888$
$55 \times 0·1638$	..	..	9·0090
$10 \times 0·0027$	..	..	0·0270
			38·5248

Deducting $38^s.5248$ from $3^h\ 55^m\ 10^s$, the corresponding interval in mean solar time is $3^h\ 54^m\ 31^s.4752$.

The hour angle of the mean sun, denoted by H.A.M.S., measures *mean solar time* (M.S.T.), or *mean time* as it is usually called. Mean noon takes place when the H.A.M.S. is 0^h and midnight takes place 12^h later when the H.A.M.S. is 12^h.

We have seen (p. 49) that the local sidereal time at any place can be found when the sidereal time at Greenwich is known and *vice versa*, the longitude of the place being given. If the longitude is given in degrees and the decimal of a degree it is only necessary to divide the number expressing the longitude by 15 to reduce it to sidereal hours. The local sidereal time at a place A in longitude λ° is $\lambda^h/15$ less than that at Greenwich, and similarly, the local solar time at A is $\lambda^h/15$ less than that at Greenwich. (See (13) p. 49).

Zone Times

If local solar times were observed throughout a country great inconvenience would result from the arrangement, as a person travelling eastward or westward would require to adjust his watch very frequently. Instead of observing local time it is usual to adopt a legal time which depends on a standard meridian—in England this is Greenwich. In the case of a ship at sea the earth's surface is divided into zones bounded by meridians of longitude which are 1^h apart, and inside a zone the mean time of the central meridian is kept. Thus, in the zone between the meridians of $\frac{1}{2}^h$ W. and $1\frac{1}{2}^h$ W., the meridian of 1^h W. is used; between the meridians of $1\frac{1}{2}^h$ W. and $2\frac{1}{2}^h$ W. the meridian of 2^h W. is used; between $2\frac{1}{2}^h$ W. and $3\frac{1}{2}^h$ W. the meridian of 3^h is used and so on. These are designated zones 1, 2, 3, etc., but if they are in longitudes east of Greenwich they are designated -1, -2, -3, etc.

A procedure similar to this is adopted in large countries. Thus, Mid-European Time, which is observed by a number of European countries, is associated with the meridian 1^h E., but the boundaries of the zone are not necessarily $\frac{1}{2}^h$ E. and $1\frac{1}{2}^h$ W. In the United States of America there are five zones which are 4^h, 5^h, 6^h, 7^h and 8^h slow on Greenwich, and the times are known as Atlantic, Eastern, Central, Mountain and Pacific times respectively.

Greenwich Mean Time or Universal Time

The civil day begins at mean midnight and ends at the mean midnight following. We have seen that the G.H.A.M.S. is 12^h at mean midnight and the Greenwich mean time clock then registers 0^h. Hence

the G.H.A.M.S. differs by 12^h from the *Greenwich Civil Time* (G.C.T.) or the *Greenwich Mean Time* (G.M.T.) as it is often called, or *Universal Time* (U.T.), because the Greenwich meridian is now, by international agreement, regarded as the standard meridian. The hours are reckoned from mean midnight, which is 0^h, up to 24^h later. Thus, July $21^d\ 00^h$ means the beginning of July 21, and we might regard it as midnight of July 20, and July $21^d\ 04^h$ means 4^h after the beginning of the day which, in civil usage, is denoted by 4 a.m. When we deal with time after mean noon we add 12^h to the civil usage which is denoted by p.m. Thus, July 21, 7 p.m., is expressed as July $21^d\ 19^h$, U.T., or G.M.T. or G.C.T.

From the beginning of 1925 dates are recorded thus: 1925 March 7; 1946 June 3, etc. Prior to 1925 the method was: April 6, 1924; May 7, 1893, etc.

Before proceeding to other problems a few examples to illustrate the subject matter in the text follow.

Example 1

If the hour angle of a star at a place in longitude 8° E. is $4^h\ 08^m\ 32^s$, find its hour angle at Greenwich.

On p. 47 it has been shown that

$$\text{H.A. of a star} = \text{local sidereal time} - \text{star's R.A.}$$

and since the R.A. does not change, the change in the H.A. must be due to the local sidereal time. The local sidereal time—in this case at Greenwich—is behind that in longitude 8° E. by 8/15 sidereal hour, that is by 32^m. Hence the H.A. of the star at Greenwich is $3^h\ 36^m\ 32^s$.

Example 2

The H.A. of a star at Greenwich is $8^h\ 18^m\ 45^s$. What is its H.A. at Philadelphia, longitude 75° 16′ 45″ W.?

The longitude is easily found to be $5^h\ 01^m\ 07^s$, and hence the local sidereal time is behind that at Greenwich by the above amount. Therefore the H.A. of the star is $3^h\ 17^m\ 38^s$.

Example 3

An observation is made at Madras, longitude 80° 14′ 20″ E. on 1945 October 6, at $14^h\ 28^m\ 32^s$ mean time. What is the corresponding sidereal time?

Mean time at Madras, October 6 ..	14^h	28^m	32^s
Longitude in time, east	5	20	57
Mean time at Greenwich, October 6 .	9	07	35
Sidereal time at 0^h, October 6 (*N.A.*, p. 18)	0	57	09
Change in sidereal time in $9^h\ 07^m\ 35^s$ ..		1	30
Mean sun's R.A.	0	58	39
Mean time at Madras	14	28	32
Sidereal time	15	27	11

The change in sidereal time (see line 6 above) can be computed from (25), or ordinary tables, if available, will be more convenient. The time is given to the nearest second.

The Length of the Morning generally differs from that of the Afternoon

EXAMPLE 4

On 1945 September 25 the equation of time is $8^m\ 04^s$. What is the difference between the lengths of the morning and afternoon?

This problem will be used as a particular case of the more general case—that the length of the morning exceeds that of the afternoon by twice the equation of time. The solution of the problem should be thoroughly understood, as a certain interesting phenomenon, which has puzzled many people, depends on the above relationship between the lengths of the morning and afternoon.

If we deal with a star the interval of time from its rising to its crossing the meridian is exactly the same as the interval from the instant of crossing the meridian until it sets. This can be verified by means of a globe, or it is obvious from the formula,

$$\cos h = -\tan \phi \tan \delta.$$

The same thing is not quite true about the sun because his declination varies slightly in the course of a day, and hence δ is a little different from sunrise to noon from what it is from noon to sunset. We shall ignore this small change in the sun's declination (though the reader should notice that it exists and for extreme accuracy must sometimes be taken into account) and shall assume that the interval from sunrise to apparent noon, that is, noon as indicated by the sun, not by a clock, is the same as the interval from apparent noon to sunset. What is implied in the ordinary words "morning" and "afternoon"?

When we speak of the morning we always imply the interval between

sunrise and *mean* noon, and similarly by afternoon we imply the interval between *mean* noon and sunset. In the case under consideration the equation of time is $8^m\ 04^s$, in other words, sun time — clock time is $8^m\ 04^s$. As the sun time or apparent time exceeds the mean time it follows that apparent noon will precede mean noon by an amount equal to the equation of time.

On September 25 the sun rises about $5^h\ 50^m$* in the latitude of 52° N. the time $5^h\ 50^m$ being indicated by a clock which records mean time. Hence the time from sunrise to mean noon is $12^h - 5^h\ 50^m = 6^h\ 10^m$, and as apparent noon precedes mean noon by $8^m\ 04^s$ it follows that apparent noon takes place $6^h\ 10^m - 8^m\ 04^s = 6^h\ 01^m\ 56^s$ after sunrise.

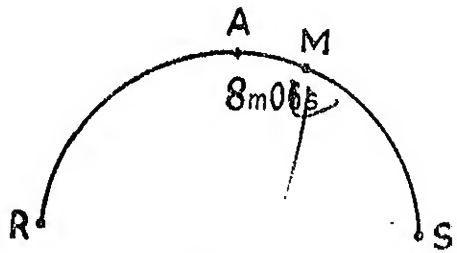

FIG. 24
Showing that the difference between the length of the morning and afternoon depends on the equation of time.

Neglecting the small changes in the sun's declination during the day, sunset takes place $6^h\ 01^m\ 56^s$ after apparent noon. But it has been shown that apparent noon precedes mean noon by $8^m\ 04^s$, and hence sunset takes place $6^h\ 01^m\ 56^s - 8^m\ 04^s = 5^h\ 53^m\ 52^s$ after mean noon. The results are as follows:

Mean noon takes place $6^h\ 10^m\ 00^s$ after sunrise.
Sunset takes place $5^h\ 53^m\ 52^s$ after mean noon.
Length of morning — length of afternoon $= 16^m\ 08^s = 2$ (equation of time) .. .. (28)

Fig. 24 illustrates this, *R* and *S* referring to the sun at rising and setting, and *A* and *M* to apparent and mean-noon.

The relation (28) always holds, regard being taken of the sign of the equation of time. If it is negative the length of the afternoon will exceed that of the morning.

* A computation by (21), gives a result differing a little from this, for reasons given in Chapter V, but this does not invalidate the results.

After the Shortest Day of the Year the Afternoons Increase in Length while the Mornings Decrease

Example 5

Explain the phenomenon noticed in the northern hemisphere that after the shortest day of the year, about December 22, the afternoons increase in length while the mornings continue to shorten. When does this phenomenon cease and what is the reason for its cessation?

When the sun is at his greatest northern or southern declination, his declination changes very slowly and hence we can consider the days as practically equal in length for a week or more. As an instance of the slow alteration in the length of the days about the time of the winter solstice when the sun is at his greatest southern declination, take the hour angles of sunrise and sunset on 1945 December 22, when the sun's declination is $-23^\circ 26' 45''$ and 1946, January 2, when it is $-22^\circ 59' 27''$.

In the latitude of Greenwich h is $24^h - 3^h 48^m 24^s$ at sunrise and $3^h 48^m 24^s$ at sunset on December 22. On January 2 following the corresponding figures are $24^h - 3^h 51^m 04^s$ and $3^h 51^m 04^s$. The equations of time on these two dates are $1^m 40^s$ and $-3^m 43^s$ respectively. Hence we obtain the following results:

Date		*Length of morning*			*Length of afternoon*		
December 22	..	3^h	50^m	04^s	3^h	46^m	44^s
January 2 ..	..	3	47	21	3	54	47

It will be seen from these figures that, while the lengths of the afternoons increase from December 22 until January 2, this increase being $8^m 03^s$, the lengths of the mornings decrease in the same time by $2^m 43^s$.

On January 6 $h = 3^h 53^m 54^s$ at sunset and the equation of time is $-5^m 34^s$, so that the length of the morning is $3^h 48^m 20^s$ and that of the afternoon $3^h 59^m 28^s$. Hence at this time the mornings have started to increase, but they are not yet as long as they were on December 22. On January 13 $h = 4^h 00^m 30^s$ at sunset, and the equation of time is $-8^m 29^s$: hence the length of the morning is $3^h 52^m 01^s$, which exceeds the length of the morning on December 22 by only 2 minutes. On the other hand, the length of the afternoon on January 13 is almost $4^h 09^m$, which is more than 22 minutes longer than the length of the afternoon on December 22.

It will be seen that the reason for the cessation of the shorter mornings is due to the increasing northern declination of the sun.

Times of Rising and Setting and of Transit of a Heavenly Body

When the declination of a body is known its hour angle of rising and setting for any specified latitude can be found by (21), but this does

not tell us anything about its actual time of rising and setting nor about its time of transit. These can be obtained if the right ascension of the body is given in addition to its declination, and the method of computation can be more easily understood from an example.

Example 6

Taking the co-ordinates of α Leonis given on p. 63, find the time that this star rises and sets and transits in the latitude of Greenwich on 1946 November 1.

The problem will be simplified if the M.S.T. of transit is first of all determined.

It has been shown that the local sidereal time at the instant of transit is equal to the star's R.A., and hence in the present case the local sidereal time is $10^h\ 05^m\ 26^s$. On referring to the *N.A.* it will be seen that the sidereal time on November $1^d.0$ is $2^h\ 38^m\ 42^s$, and hence an interval of $7^h\ 26^m\ 44^s$ ($10^h\ 05^m\ 26^s - 2^h\ 38^m\ 42^s$) has elapsed since midnight. This is reckoned in sidereal time and the corresponding interval in M.S.T. is $7^h\ 25^m\ 31^s$. Hence the G.M.T. of the transit of α Leonis is $7^h\ 25^m\ 31^s$.

To find the times of rising and setting, use (21) from which $h = 7^h\ 03^m\ 12^s$ at the time of setting and $16^h\ 56^m\ 48^s$ at the time of rising. It has been shown that

$$\text{local sidereal time} = \text{H.A. of star} + \text{star's R.A.}$$

Hence, substituting the values of the star's R.A. and H.A., it is found that the local sidereal time at the instant of rising is $16^h\ 56^m\ 48^s + 10^h\ 05^m\ 26^s = 3^h\ 02^m\ 14^s$. By the same method it is found that the local sidereal time at the instant of setting is $17^h\ 08^m\ 38^s$.

The interval in sidereal time after midnight until the star rises is $3^h\ 02^m\ 14^s - 2^h\ 38^m\ 42^s = 0^h\ 23^m\ 32^s$ and the interval after midnight until the star sets is $17^h\ 08^m\ 38^s - 2^h\ 38^m\ 42^s = 14^h\ 29^m\ 56^s$. The mean of these is $7^h\ 26^m\ 44^s$ and is the sidereal interval from midnight until the time of transit. This corresponds with the value found earlier and is a check on the accuracy of the work.

The sidereal time interval of $0^h\ 23^m\ 32^s$ corresponds to a mean time interval of $0^h\ 23^m\ 28^s$, and the sidereal time interval of $14^h\ 29^m\ 56^s$ corresponds to a mean time interval of $14^h\ 27^m\ 34^s$. Hence the results are as follows:

G.M.T. of rising of α Leonis	0^h	23^m	28^s
G.M.T. of transit	7	25	31
G.M.T. of setting	14	27	34

It will be observed that the time of transit is exactly the mean of the times of rising and setting.

If we require the times of rising and setting and of transit approximately—say to within two or three minutes, which is sufficiently accurate for many practical purposes—it is unnecessary to make the corrections for changing sidereal intervals into mean time intervals. The method in this case is as follows.

Having found the time of transit and also the hour angle of setting (which will prove more convenient than the hour angle of rising), the results are

Time of transit	..	..	7^h	29^m	27
H.A. of setting	..	..	7	03	
Time of setting	..	..	14	32	30
Time of rising	..	..	0	26	24

The figures are given to the nearest minute and no distinction is drawn between sidereal time intervals and mean time intervals. As will be seen later, no corrections have been applied in the accurate computations for refraction, and as this would make a difference of a few minutes in the times of rising and setting, none of the figures shown in the first computations are really quite correct.

Twilight

We shall conclude this chapter by dealing with the phenomenon of twilight, which is due to the light of the sun being scattered or reflected in various directions when the sun is below the horizon. *Astronomical twilight* is said to end when the sun is 18° below the horizon, and *nautical twilight* is considered to end when the sun is 12° below the horizon. There is another twilight known as *civil twilight*, which ends when the sun is 6° below the horizon. Problems involving twilight can be solved by (16), as the following examples will show.

Suppose we want to find the latitude of a place at which twilight will just last all night, it is only necessary to make $z = 90° + 18°$, or 108° and $h = 12^h$ or 180°, because the sun attains its greatest distance below the horizon when $h = 12$ (see p. 38), and if this distance is 108° twilight will be in evidence then. At an earlier or later time than is determined from $h = 12^h$ the sun will not be so far below the horizon, and hence twilight will last all night.

Since $\cos(90° + 18°) = -\cos 72°$, and $\cos h = -1$, (16) becomes

$$-\cos 72° = \sin\phi \sin\delta - \cos\phi\cos\delta = -\cos(\phi + \delta)$$
$$\text{Hence } \phi + \delta = 72°.$$

Suppose we wanted to find the conditions that twilight should last all night in latitude ϕ, we have $\phi + \delta = 72°$, or $\delta = 72° - \phi$. If the sun

should be 19° below the horizon twilight would not last all night and in this case $\phi + \delta = 71°$, from which it is seen that if $\phi + \delta < 72°$, twilight will not last all night. Hence the sun's declination must be just greater than $72° - \phi$, and if $\phi = 51\frac{1}{2}°$ N. the declination must exceed $+ 20\frac{1}{2}°$. On referring to the *N.A.* it will be found that the sun attains this declination about May 23 and after this twilight will last all night at latitude $51\frac{1}{2}°$ N. until July 21, when the sun again attains a declination of about $+ 20\frac{1}{2}°$. After this his declination is less than $20\frac{1}{2}°$ and twilight will not last all night until May 23 of the following year.

Example 7

Find the duration of civil twilight on October 19 when the sun's declination is about $- 10°$, the latitude of the place being 48° N.

Two problems are involved here. First of all it is necessary to find the hour angle of the sun at setting; then we must find the hour angle of the sun when he is 6° below the horizon, and the difference between these will give the duration of twilight, because civil twilight lasts from the time the sun sets until the time that he is 6° below the horizon.

From (21) $h = 78° 42'$. To find h in the second case, substitute 96° for z in (20) and solve for h. This gives us the expression

$$\cos h = (\cos 96° - \sin 48° \sin{-10°})/(\cos 48° \cos{-10°}) = 0{\cdot}0894.$$

Hence $h = 84° 52'$, and the difference between the two hour angles is $6° 10'$, which is equivalent to nearly 25 minutes. The duration of civil twilight under the above conditions is, therefore, 25 minutes after sunset. The next morning it would last about 25 minutes before sunrise.

Example 8

Find the duration of twilight at the equator at the equinoxes.

In this case both ϕ and δ are 0° and (20) reduces to the simple form

$$\cos h = \cos 108°, \text{ and hence } h = 108°.$$

If $z = 90°$, which occurs at sunrise or sunset, then (21) becomes

$$\cos h = - \tan \phi \tan \delta = 0, \text{ or } h = 90°.$$

Hence twilight after sunset or before sunrise will last 18/15 hours, or $1^h\ 12^m$.

Example 9

Find the duration of nautical twilight at the equator at the solstices.

In this case $\delta = \pm 23° 27'$, and h is easily found to be $180° - 76° 54' = 103° 06'$ at sunset and $180° + 76° 54' = 256° 54'$ at sunrise.

The hour angles at sunset and sunrise are 90° and 270° respectively, and hence nautical twilight lasts after sunset or before sunrise for nearly $52\frac{1}{2}$ minutes, the equivalent of 13° 06′.

Detailed working of the above examples has not been shown. The reader should be able to check the figures obtained by using four-figure logs. In the problems which follow four figures are all that will be required.

Problems

1. On January 17 the equation of time is -10^m. By how much does the afternoon exceed the morning?

2. Express the mean time intervals of (a) $14^h\ 50^m$, (b) $17^h\ 53^m\ 10^s$, (c) $3^h\ 12^m\ 57^s$ as intervals of sidereal time.

3. If twilight just lasts all night when the sun's declination is $-20°$, what is the latitude of the place?

4. On December 1 a person wants to find his south by the transit of the sun. The equation of time on this date is $11^m\ 09^s$, and his watch records correct mean time. At what time should he take the sun's bearings to find the south?

5. Use (21) to find the times of rising and setting of the sun at a place in latitude 74° N. when the sun's declination is $+22°$. What interpretation can be placed on the result?

6. If twilight lasts all the night in latitude 62° N. what are the limits of the sun's declination?

7. What is the duration of twilight (nautical) at the latitude of Greenwich on November 5 when the sun's declination is $-15°\ 30'$?

8. What are the altitude and azimuth of α Geminorum on February $20^d\ 21^h\ 30^m$ U.T. at a place in longitude 12° E. and latitude 50° N.? (The sidereal time at Greenwich on February $20^d\ 00^h$ can be taken as 10^h.)

CHAPTER V

ATMOSPHERIC REFRACTION

A RAY of light moves through a transparent medium in a straight line only so long as the density of the medium remains uniform. If the ray passes obliquely from one medium to another its course will be bent at the point of incidence. Two important conditions are fulfilled when a ray of light is thus bent or refracted: first, the two directions before and after incidence will lie in the same plane with the perpendicular or *normal* (as it is usually described) to the surface at that point; second, the sines of the angles formed by the directions of the ray with the normals are in a constant ratio.

When a ray of light passes from a rarer into a denser medium its direction is altered in such a way that it approaches the normal, as shown in Fig. 25, but if the ray enters the medium at right angles to its surface, that is, parallel to the normal, refraction does not take place. In Fig. 25 if i is the angle between the ray from the candle and the normal, and r is the angle between the direction of the refracted ray and the normal, then

$$\sin i/\sin r = \mu \qquad \ldots \quad \ldots \quad \ldots \quad (29)$$

where μ, a constant depending on the medium, is known as the *refractive index* of the medium.

Fig. 26 shows how a ray of light from a celestial body is refracted by the atmosphere, the ray being bent towards the normal because, as it approaches the earth's surface, it gradually passes through strata of the atmosphere of increasing density. The figure shows why objects appear higher than they actually are, and why refraction must be taken into consideration in dealing with astronomical problems where great accuracy is required.

Several formulae have been derived for computing the atmospheric refraction of light from a body, but it will be sufficient if the method for deriving one of these is given.

There is a law known as the *law of successive refractions* which can be stated thus:

If there be a number of different media separated by parallel planes and a ray of light pass through these media, suffering refraction at their boundaries, the final direction of the ray is parallel to what it would

have been if the ray had been refracted directly from the first into the last medium without passing through the intermediate media.

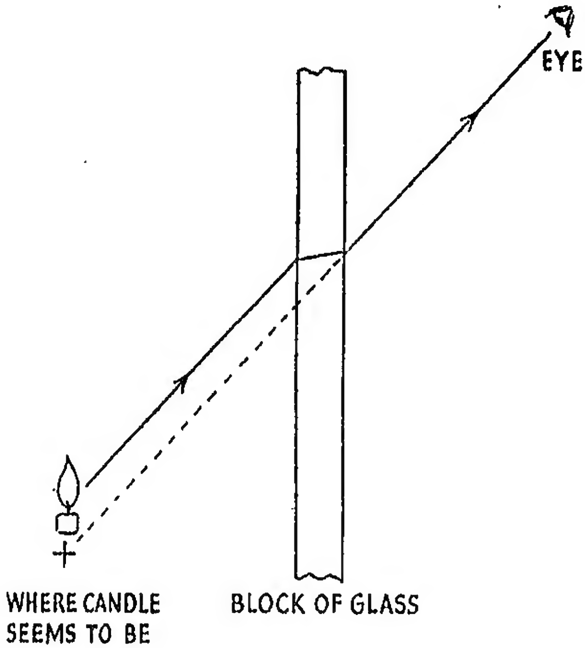

Fig. 25
Refraction of a ray of light.

This law can be easily proved from elementary geometrical considerations, but it will be assumed to hold, though it should be emphasized that it holds only in the case of *parallel planes*. Remembering that the height of the earth's atmosphere is very small relative to the earth's

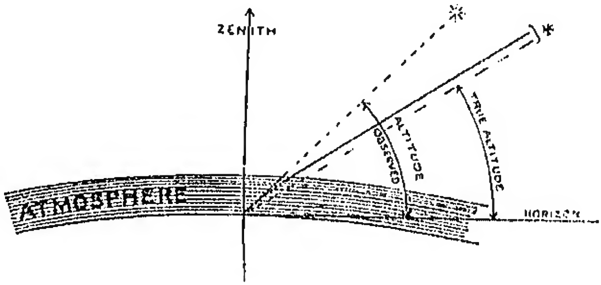

Fig. 26
Atmospheric refraction of light. On account of this, celestial objects appear higher than they actually are.

radius, it is obvious that we can regard the small portion of the earth's surface with which we are dealing as flat and the successive atmospheric strata as parallel, so that the law holds almost exactly in these circumstances. We may therefore consider that a ray of light from a celestial body enters the atmosphere at a height of about 50 miles, where the density is so small that we can, for the present purpose, regard it as a vacuum, and reaches the earth's surface by a single refraction, the intermediate strata of the atmosphere being ignored.

In Fig. 27 the atmosphere is represented as consisting of a number

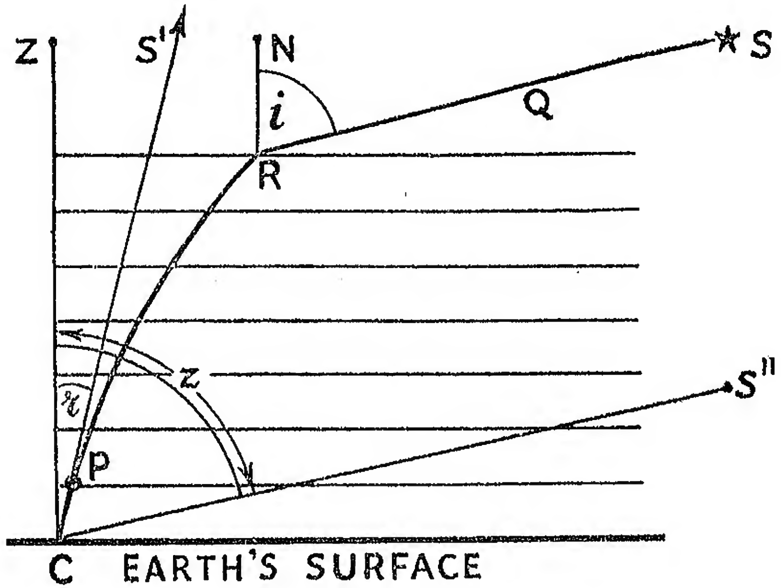

FIG. 27
Derivation of a formula for atmospheric refraction.

of horizontal layers in which the density increases towards the earth's surface, the index of refraction also increasing. A ray QR from a star S reaches the highest layer at R, the angle of incidence being i, RN being the normal, and is refracted so that RPC is its path in the atmosphere, PC being the last short portion of its path. An observer at C on the earth's surface sees the star in the direction CPS', and if CZ is the direction of the zenith the angle ZCS' is the star's *observed zenith distance.*

If CS'' is drawn parallel to QR, $\widehat{ZCS''}$ is the *true zenith distance* of the star, because the star can be considered at an infinite distance. It should be noticed that $\widehat{ZCS''}$ would not be the true zenith distance if we were dealing with a close body, but even in the case of the moon the

error introduced by this assumption is so small that for all practical purposes it can be ignored.

Let the true zenith distance ZCS'' be z and let the observed zenith distance ZCS', which is marked r in the figure and corresponds to r in (29), be ζ. Since $\sin i = \mu \sin r$, and $\widehat{ZCS''} = i$ (because QR and the normal at R are parallel to CS'' and CZ, respectively), then, μ being the index of refraction of the lowest layer, by (29),

$$\sin z = \mu \sin \zeta$$

The angle $S'CS''$ through which the star's zenith distance is displaced is known as the *angle of refraction* and is denoted by R. From the figure it is seen that

$$R = z - \zeta,$$

hence,

$$\sin (R + \zeta) = \mu \sin \zeta \quad \ldots \quad \ldots \quad \ldots \quad (30)$$

The left-hand side of (30) can be expressed in the form

$$\sin R \cos \zeta + \cos R \sin \zeta.$$

But R is always a small angle and hence, expressed in radian measure,

$\sin R = R$, $\cos R = 1$, and the above becomes

$$R \cos \zeta + \sin \zeta,$$

from which, $R \cos \zeta + \sin \zeta = \mu \sin \zeta$, or

$$R = (\mu - 1) \tan \zeta \quad \ldots \quad \ldots \quad \ldots \quad (31)$$

In (31) R is expressed in radians, but it is more convenient to express it in seconds of arc, one radian being 206,265 seconds of arc. Hence (31) can be expressed in the form

$$R = 206{,}265\ (\mu - 1) \tan \zeta,$$
$$\text{or } R = k \tan \zeta \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (32)$$

In the last formula k depends on the value of μ at the earth's surface and is known as the *constant of refraction*. Its value, derived from observation, is 58".2 when the height of the barometer is 30 inches, and the temperature is 50°F. Hence (32) can be written in the form

$$R = 58''.2 \tan \zeta \quad \ldots \quad \ldots \quad \ldots \quad (33)$$

The *mean refraction* R, given by (33), enables us to calculate the true zenith distance z since this is $R + \zeta$, and ζ is obtained from observation.

In cases where the barometric pressure and temperature differ from those just given, the correction to R can be made by means of the formula

$$R_1 = \frac{17BR}{460 + T} \qquad \text{.. .. ..} \qquad (34)$$

where R_1 denotes the refraction when the height of the barometer is B inches and T is the temperature in degrees on the Farenheit scale.

When $\zeta = 90°$, R is infinite, which is absurd, and in fact (33) cannot be used for zenith distances beyond about 70°. Other formulae must be used in cases where ζ exceeds 70°, and for zenith distances close to 90° it is impossible to derive the refraction by any practicable formula. When ζ is 90°, that is, when the body is on the horizon, the refraction, then known as the *horizontal refraction*, is 34′, this value being drived from observation. When $\zeta = 0°$, that is, when the body is in the zenith, $R = 0$, or there is no refraction, a result previously referred to when it was stated that refraction does not take place when a ray of light enters a medium at right angles to its surface.

We shall illustrate the formulae derived by two examples.

Example 1

A star is observed at an altitude of 60°. What is its true altitude standard atmospheric conditions being assumed?

The observed zenith distance ζ is $90° - 60° = 30°$, and hence

$$R = 58''\!.2 \tan 30°$$

$\log 58{\cdot}2$	$1{\cdot}7649$
$\log \tan 30°$	$\bar{1}{\cdot}7614$
$\log R$	$1{\cdot}5263$
R	$33''\!.6$

The true zenith distance is 30° 00′ 33″.6, and hence the true altitude is 59° 59′ 26″.4.

Example 2

In the last example what would be the true altitude of the star if the barometer stood at 29 inches and the temperature were 60°F.?

$$B = 29,\ T = 60°.$$

Using (34) the results are as follows:

log 17	1·2304
log 29	1·4624
log R	1·5263
sum	4·2191
log 520	2·7160
log R_1	1·5031
R_1	31.85

$z = 30° \ 00' \ 31''.85$, and the true altitude is $59° \ 59'' \ 28''.15$.

Some Effects of Refraction

Amongst the many effects of refraction may be noticed that on the rising and setting of heavenly bodies. A star can be seen on the horizon, assuming ideal seeing conditions, when it is actually 34′ below the horizon, and it does not set until it is 34′ below the horizon. Hence the rising of a heavenly body is hastened and its setting is retarded by refraction, and in the formulae used for finding the hour angles of its apparent rising and setting, z must be made equal to 90° 34′ in (16). By doing so A, derived from (17) or (18), will always be placed more to the north in the northern hemisphere than it would be if there were no atmosphere.

The Sun is Considered to Rise and Set when his Upper Limb is on the Horizon

When we are dealing with the hour angle of the rising or setting sun or with the azimuth of the sun at rising or setting, another correction must be made. The sun is considered to rise *when his upper limb is on the horizon* (the same applies to the moon), and since the radius of the sun subtends an angle of about 16′ at the earth, the centre of the sun is $34' + 16' = 50'$ below the horizon at sunrise and sunset. Hence z must be made equal to 90° 50′ in (18), and in other equations where it has been previously taken as 90°, when we are dealing with the sun. An example will show the effect of these two corrections, which, though small, must nevertheless be taken into consideration when accuracy is required.

EXAMPLE 3

In the example given on p. 61 in which the hour angle of the sun at rising and setting was found for a place in latitude 50° N. when the sun's declination was + 18°, what are the accurate figures, refraction

being taken into account and the sun being considered to rise and set when his upper limb appears on the horizon?

In this case $\cos z = \cos 90^\circ 50' = -0{\cdot}0146$, and the formula becomes

$$-0{\cdot}0146 = \sin 50^\circ \sin 18^\circ + \cos 50^\circ \cos 18^\circ \cos h$$

log cos 50°	$\bar{1}{\cdot}8081$	log sin 50°	$\bar{1}{\cdot}8843$	
log cos 18°	$\bar{1}{\cdot}9782$	log sin 18°	$\bar{1}{\cdot}4900$	
sum	$\bar{1}{\cdot}7863$	sum	$\bar{1}{\cdot}3743$	
		sin 50° sin 18°	$-0{\cdot}2368$	
		cos z — sin 50° sin 18°	$\cancel{-0{\cdot}2514}$	$+\cdot 2222$
		log $\cancel{-0{\cdot}2514}$ $\cdot 2222$	$\cancel{\bar{1}_n{\cdot}4004}$	$\bar{1}{\cdot}3468$
		deduct	$\bar{1}{\cdot}7863$	
		log cos h	$\cancel{\bar{1}_n{\cdot}6141}$	$\bar{1}{\cdot}5605$

$\cancel{h = 180^\circ \pm 65^\circ 43' = 7^h 37^m 08^s \text{ or } 16^h 22^m 52^s}$

$h \quad 68^\circ 41'$ or $360 - 68^\circ 41^m$ $\quad 4^h 34^m 44^s$ or $19^h 25^m 16^s$

It will be seen on referring to p. 61 that there is a difference of 6 minutes between the times of rising and setting of the sun in the two cases.

Example 4

Find the azimuth of the sun at rising and setting on June 21 at a place in latitude 51° 30′ N., taking refraction into consideration and assuming rising and setting to occur when the upper limb of the sun appears on the horizon.

As the reader is familiar with the use of logarithms by this time there will be no necessity to work out all the examples in full. In most cases a computing machine has been used for the calculations, but these can be checked by logarithms if the reader wishes to do so.

Substituting 90° 50′ for z, the azimuth A is found from (18).

$$\cos A = (\sin 23^\circ 27' - \sin 52^\circ 30' \cos 90^\circ 50')/(\sin 90^\circ 50' \cos 52^\circ 30'.)$$

Substituting the values for the various functions,

$$\cos A = 0{\cdot}6575, \text{ or } A = 48^\circ 54' \text{ E or W.}$$

Notice that $\sin 90^\circ 50' = 1$ in 4 fig. computation.

How Refraction Affects the Shape of the Sun and Moon

One effect of refraction is to make the sun about the time of sunrise or sunset appear oval. The horizontal refraction is 34′ and the refraction for an object with zenith distance less than 90° is less than 34′. Hence the sun's lower limb appears raised towards the zenith slightly more than his upper limb when he is just above the horizon, but refraction will not affect the sun's horizontal diameter much because each

end of it has the same zenith distance. For this reason the sun appears slightly oval when very close to the horizon, and the same applies also to the moon. The contraction of the vertical diameter amounts to about 5′. This effect has nothing to do with the apparent increase of size of the sun and moon when they are close to the horizon; this is a psychological effect and is quite independent of refraction.

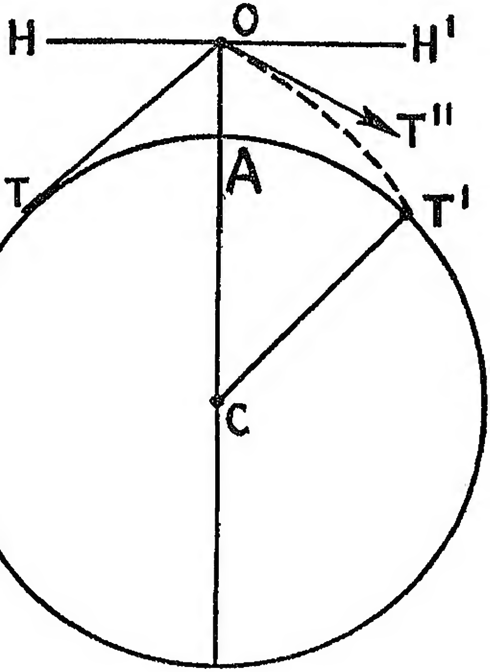

FIG. 28
The distance of the visible horizon is increased by refraction.

Another effect of refraction is to increase the distance of the visible horizon and to decrease the dip of the horizon. It has been shown that $d = 1{\cdot}064 \sqrt{h}$ and $h = 0{\cdot}883\, d^2$ (see p. 21), but owing to refraction these formulae are modified slightly, d and h being determined from

$$d = 1{\cdot}15 \sqrt{h}, \quad h = 0{\cdot}756\, d^2 \qquad \ldots \qquad \ldots \qquad (35)$$

The reason for this modification can be seen from an inspection of Fig. 28. O is an observer at a height h above the earth's surface, and the tangent OT to the sphere determines the limit of visibility if there were no refraction. But since the ray of light proceeding from T to O is

passing through strata of decreasing density, the ray will be bent towa the earth's surface. If T' be another point at a greater distance fr O than is T, the broken line shows the path of a ray of light from and as it is bent towards the earth's surface, it will strike the observ eye at O, so that he will see the horizon in the direction OT'' and it v appear at a greater distance from him than does T. The dip of visible horizon is $H'OT''$, which is obviously less than the dip HOT wl refraction is ignored.

The dip is equal to the angle OCT' because OH' and OT' perpendicular to OC and CT' respectively. Hence the dip in rad measure is the arc AT' divided by the radius of the earth. As the AT' is practically the same as the distance d, the radian measure of dip is d/r, or, expressed in seconds of arc, the dip is $206{,}265d/r$. Si r is 3442 nautical miles the dip is $60d$ seconds of arc.

Measurement of the Constant of Refraction

Different methods are in use for determining k in (32). A sim method consists in measuring the observed zenith distances of a sta upper and lower culmination, and as it is possible by this methoc determine the declination of a star and also the latitude of the p of observation, an example will be given from which the reader will more easily how the general formula is arrived at.

On p. 44 it was shown that the meridian altitudes of δ Drac in the latitude of Birmingham, 52° 59′ N., were 75° 25′ and 30° 3: upper and lower culmination respectively. In the computation effects of refraction were ignored, and it remains to deal with the prob when these are taken into consideration.

The problem will now be stated in the following form:

Example 5

The declination of δ Draconis is known to be 67° 34′ approxima and the latitude of Birmingham can be taken to be 52° 59′ N. exactly. U normal conditions of temperature and barometric pressure the meri altitudes of δ Draconis at upper and lower culminations were obse to be 75° 25′ 15″.14 and 30° 34′ 38″.37 respectively. Find the consta refraction.

Using the figure on p. 44, the results are as follows:

$$90^\circ - 67^\circ\ 34' + 52^\circ\ 59' = 75^\circ\ 25'\ 15''.14 - k \tan 14^\circ\ 3$$
upper culmination.

$$52^\circ\ 59' - (90^\circ - 67^\circ\ 34') = 30^\circ\ 34'\ 38''.37 - k \tan 59^\circ\ 2$$
lower culmination.

There is no necessity to use the seconds of arc in the tangents of the zenith distances because k is so small that k tan z will be unaffected by the seconds of arc in z.

Adding these two equations we obtain

$$k\,(\tan 14^\circ\,35' + \tan 59^\circ\,26') = 105^\circ\,59'\,53''\!.51 - 105^\circ\,58'\,00''\!. = 1'\,53''\!.51.$$

Substituting the values of tan $14^\circ\,35'$ and tan $59^\circ\,26'$, we obtain

$$1{\cdot}9533\,k = 113''\!.51, \text{ from which}$$
$$k = 57''\!.9$$

This is close to the result usually adopted, $58''\!.2$.

It will be seen that the declination of the star disappears by this method and so it is unnecessary to know its declination. In fact, it is possible in this way to find the declination of δ Draconis, or of other circumpolar stars, by substituting the value derived for k in either of the equations above. If we take the first of these we find as follows:

Let δ be the declination of δ Draconis, and ϕ and ζ the latitude of the place and the apparent zenith distance of the star. Then the first of the above equations can be expressed as follows:

$$90^\circ - \delta + \phi = 90^\circ - \zeta_1 - k \tan \zeta_1$$

or

$$\delta = \phi + \zeta_1 + k \tan \zeta_1.$$

Substituting the values of ϕ, ζ_1, and also that of k just derived, we have

$$\delta = 52^\circ\,59''\,00'' + 14^\circ\,34'\,44''\!.86 + 15''\!.08 = 67^\circ\,33'\,59''\!.94.$$

It is also possible to obtain the latitude of a place by this method even if the declination of a star is not known. The second equation can be written as follows:

$$\phi - (90^\circ - \delta) = 90^\circ - \zeta_2 - k \tan \zeta_2$$

or

$$\delta = 180^\circ - \phi - \zeta_2 - k \tan \zeta_2.$$

Subtracting this equation from the previous one to eliminate δ,

$$2\,\phi = 180^\circ - (\zeta_1 + \zeta_2) - k\,(\tan \zeta_1 + \tan \zeta_2)$$

Making the following substitutions:

ζ_1 ..	14°	34′	44″.86	$\tan \zeta_1$	0·2602
ζ_2 ..	59	25	21·63	$\tan \zeta_2$	1·6920
$\zeta + \zeta_2$ ..	74	00	06·49	$\tan \zeta_1 + \tan \zeta_2$.	1·9522
				$k (\tan \zeta_1 + \tan \zeta_2)$	113″.38

$$2\phi = 180° - 74° \, 00' \, 06''.49 - 0° \, 01' \, 53''.38 = 180° - 74° \, 01' \, 59''.87$$
$$= 105° \, 58' \, 00''.13.$$
$$\text{Hence } \phi = 52° \, 59' \, 00''.06 \text{ N.}$$

In this example the star culminates in each case north of the zenith, and in these circumstances $\delta > \phi$. A general formula has not been given because this would not hold if $\delta < \phi$, and readers are advised to work out each case independently from a suitable diagram.

Before concluding this chapter a problem on the rising and setting of the sun will be dealt with, and the reader is recommended to verify some of these problems, which he can set for himself, from the *Nautical Almanac*, as the practice will make him familiar with certain points discussed in this and the preceding chapter.

Example 6

In the *N.A.* for 1945, p. 12, the declination of the sun is given for May 17·0 as $+ 19° \, 11' \, 39''.8$ and for May 18·0 as $+ 19° \, 25' \, 13''.2$. Assuming that the declination on May $17^d \, 12^h$ is the mean of these two, that is, $+ 19° \, 18'$ to the nearest minute, and that this value is sufficiently accurate to find the hour angle of the sun at rising and setting, find the times of the rising and setting of the sun at a place in latitude 52°N., and check the results from the Tables given in the *N.A.*, p. 476.

Substituting the values for $\cos z$, etc., in (16), remembering that $\cos z$ is $- 0·0146$, the formula reduces to

$$- 0·0146 = 0·2604 + 0·5811 \cos h$$

Hence $\cos h = - 0·4732$, or $h = 180° + 61° \, 46' = 16^h \, 07^m \, 04^s$ at the time of sunrise. Hence sunrise takes place at $4^h \, 07^m \, 04^s$ and sunset at $19^h \, 52^m \, 56^s$ by sun time. The equation of time on May 17·0 is $3^m \, 44^s.86$ and on May 18·0 it is $3^m \, 43^s.28$, and we can take $3^m \, 44^s$ as sufficiently accurate for the present purpose. Applying (24)

clock time = sun time − equation of time, we have

clock time = $4^h \; 07^m \; 04^s - 3^m \; 44^s = 4^h \; 03^m \; 20^s$ = time of sunrise.

$19 \; 52 \; 56 - 3 \; 44 = 19 \; 49 \; 12$ = time of sunset.

The *N.A.* gives the times to the nearest minute in all cases and for May 17 they are given as $4^h\ 04^m$ for sunrise and $19^h\ 49^m$ for sunset—results in sufficiently good agreement with those just derived.

Problems

(The barometer is assumed to be at a height of 30 inches and the temperature to be 50°F., unless otherwise stated.)

1. The apparent altitude of a star is $60^\circ\ 32'\ 45''.80$. What is its true altitude?

2. At an observatory in the northern hemisphere the observed zenith distances of a star at upper and lower culmination are $7^\circ\ 22'\ 11''.89$ and $69^\circ\ 37'\ 47''.13$ respectively. The upper culmination is north of the zenith. Find the latitude of the observatory and the star's declination.

3. A man looks out to sea from the top of a tower 180 feet above sea level. How far can he see (a) neglecting refraction; (b) taking refraction into consideration?

4. Find the dip of the visible sea horizon when the eye is 200 feet above sea level, (a) when refraction is ignored; (b) when it is taken into consideration.

5. Find the time of sunrise and sunset on July 1, taking the sun's declination to be $+23^\circ\ 07'$ and the equation of time to be $-3^m\ 36^s$, at places in latitudes (a) 60°; (b) 55°; (c) 50° N.

6. In example 5 what are the Greenwich times of sunrise and sunset if the longitudes of the places are (a) 1° E.; (b) 1° W.; (c) 1° 15′ W.?

7. To what latitude would you require to go on June 1 when the sun's declination is about $+22^\circ$ so that there would be no sunset?

8. The true altitude of a star is $50^\circ\ 24'\ 32''$. What is its apparent zenith distance?

9. In example 8 if the barometric height is 29·1 inches and the temperature of the atmosphere is 35°F., what is the star's apparent zenith distance?

10. What is the sun's azimuth at rising and setting in a place where the latitude is 48° N. on June 23 when the sun's declination is $+23^\circ\ 27'$? What are the corresponding figures for December 22 when the sun's declination is $-23^\circ\ 27'$?

11. Find the times of rising and setting of the sun on December 22 and January 2 in latitude $52\frac{1}{2}^\circ$ N. (Equation of time to be taken into consideration.)

Chapter VI

PARALLAX

If you want a good illustration of parallax hold a finger in front of your eyes and look at a distant object, closing each eye in turn. You will notice that your finger appears to be displaced to the right if your left eye is open and to the left if your right eye is open, and also that the closer you hold your finger to your eye the greater the displacement seems to be. For the distant object substitute one of the remote stars—so far away from the earth that it may be considered at an infinite distance; for your finger substitute a comparatively close celestial body, like the moon; and for each eye in turn imagine that you are looking at the moon from two places separated—not by three inches as in the case of your eyes—but by thousands of miles. Just as your finger appeared to be displaced when viewed by each eye in turn, this displacement taking place with reference to some distant object, so the moon and other relatively close bodies appear displaced with reference to the background of stars if they are viewed from different places on the surface of the earth.

Some of the problems previously dealt with related to the sun whose declination was given for a certain time, but nothing was said about the place on the earth from which the declination was measured. It would obviously be inconvenient if the *N.A.* had to supply the declination of the sun, moon, and planets for every observatory in the world, and so these co-ordinates are always given for an observer at the centre of the earth. Of course there is no such thing as an observer in this position, but we can imagine that the earth is transparent and that someone at its centre can see the heavenly bodies and measure their positions. Actually, when these positions are measured from any observatory, it is a simple computation to make the necessary reductions and to calculate what their right ascensions and declinations or longitudes and latitudes would be if viewed from the earth's centre.

The parallax of a heavenly body is the angle between two lines drawn to it, one from the observer, wherever he may be on the earth's surface, and the other from the earth's centre. (This applies only to comparatively close bodies—like those in the solar system.) This angle is small in the case of the sun and fairly large when we are dealing with the moon, but it cannot be detected by the most delicate instruments when we are dealing with the stars. The reason for the failure to detect this

angle in the case of the stars will be evident from the following considerations.

The nearest star to us is about 25×10^{12} miles away, and the greatest distance between two observatories on the earth is about 8,000 miles—the earth's diameter. The earth's diameter subtends one second of arc at a distance of about 16×10^{8} miles and hence it would subtend $0''.000064$ at the distance of the nearest star—an angle utterly impossible to measure by the most delicate instrument. Owing to the enormous

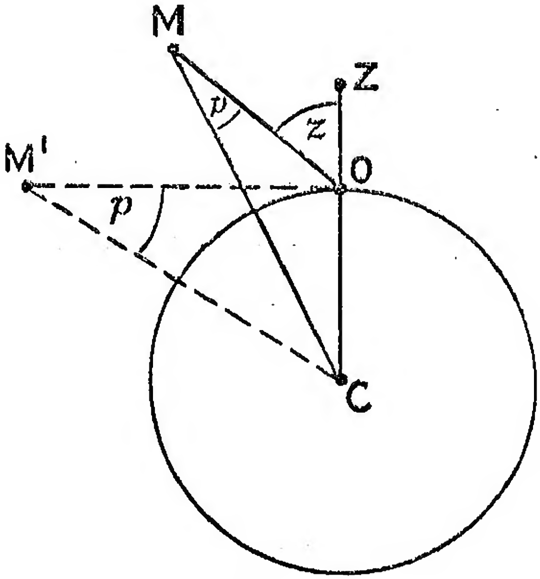

Fig. 29
Derivation of parallax formula.

distances of the stars in comparison with the bodies of the solar system they can be used as the background or points of reference when the parallaxes of closer bodies are to be determined.

Derivation of Parallax Formulae

In Fig. 29 C is the centre of the earth and O an observer on its surface, M being a celestial body, say the moon. CO produced will pass through the zenith of O, and the effect of parallax will be to change the zenith distance of M from $\widehat{ZCM}$ to $\widehat{ZOM}$. The angle CMO is the parallax, and since $\widehat{ZOM}$ is equal to the sum of the angles ZCM and CMO, the parallax CMO is the difference between the two zenith distances. The angle ZCM is known as the *true zenith distance.*

Let a be the earth's* radius, d the distance from the centre of the earth to M, p the parallax CMO, and z the zenith distance ZOM. From the elementary properties of the plane triangle COM,

$$\sin p/\sin z = a/d, \text{ hence,}$$

$$\sin p = \frac{a}{d} \sin z \qquad \ldots \quad \ldots \quad \ldots \quad (36)$$

(Notice that $\sin \widehat{COM}$, which is the angle considered in the triangle COM, is the same as $\sin (180° - z) = \sin z$.)

When the body is on the horizon, $z = 90°$ and p becomes the *horizontal parallax* which will be denoted by P. (36) then reduces to

$$\sin P = a/d \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (37)$$

This is otherwise obvious from the figure in which M' represents the moon on the horizon, so that the angle COM' is 90°. Hence

$$\sin \widehat{CM'O} = \sin P = CO/CM' = a/d.$$

By (36) $a/d = \sin p/\sin z$, and combining this with (37)

$$\sin p = \sin P \sin z \qquad \ldots \quad \ldots \quad \ldots \quad (38)$$

In the above investigation it is assumed that the effect of refraction has been removed from the observed zenith distance so that $\widehat{ZOM}$ is the *apparent zenith distance* derived by (33).

There is a simpler form for (37) which can be used in all cases, even when we are dealing with the moon, the nearest celestial body to us, and hence the parallax of which is greater than it is for any other heavenly body. This derivation of this form is easily verified by substituting 4000 for a and 240,000 for d in (37), so that $\sin P = 1/60$. The angle whose sine is 1/60 is 0° 57′ 17″.90 and the angle whose radian measure is 1/60 is 0° 57′ 17″.75, the difference being only 0″.15. Hence we can write $\sin p = p$ and $\sin P = P$, where p and P are in radian measure, without any serious error in the case of the moon, the nearest celestial body to us, and *a fortiori* in the case of the sun and planets. In (38) $\sin p/\sin P = \sin z$, and since $\sin p/\sin P$ is the same as p/P, where p and P are in either radian measure or seconds of arc, the *ratio* remaining unchanged if seconds of arc are substituted for the radian measure, we obtain the simple expression,

$$p'' = P'' \sin z \qquad \ldots \quad \ldots \quad \ldots \quad (39)$$

* Parallaxes are always expressed in terms of the earth's *equatorial* radius.

Also, (37) can be written in the form

$$P = 206{,}265'' a/d \qquad \ldots \quad \ldots \quad \ldots \quad (40)$$

The moon does not move round the earth in a circle but in an ellipse, so that d varies and hence P varies also. The *N.A.* supplies the value of P for the moon for every day of the year, and from this p can be computed by (39).

From the method for deriving the above formulae it is seen that the azimuth of a body is not affected by parallax. Only the zenith distance (or altitude) is affected, and the zenith distance is increased, contrary to the effect of refraction, which decreases the zenith distance.

The horizontal parallax of the moon is about 1°, and as the sun is nearly 400 times as far from the earth as is the moon, the horizontal parallax of the sun is about 1/400 of a degree or about 9 seconds of arc. In some of the examples which follow it can be taken as $8''.79$, but as the parallax of the moon varies much more than does that of the sun, it will be given for the particular time of the observation.

A few examples illustrate the application of the formulae just given. It is assumed that the effect of refraction has been removed so that the correction for parallax only is to be applied.

Example 1

The sun's observed zenith distance is 35°. Find his true zenith distance.

$$8''.79 \sin 35° = 8''.79 \times 0{\cdot}5736 = 5''.04.$$

Hence the sun's true zenith distance is $35° - 5''.05 = 34°\ 59'\ 54''.95$.

Example 2

If the moon's horizontal parallax is $60'.2$, find her true zenith distance if her observed altitude is 30°.

$$z = 60°,\ \sin z = 0{\cdot}8660,\ 60'.2 \times 0{\cdot}8660 = 52'.13.$$

Hence the moon's true zenith distance is $60° - 52'.13 = 59°\ 07'\ 52''$.

Example 3

If the moon's horizontal parallax is 60′ what must be her zenith distance so that the correction to apply for parallax is 45′?

If z be the required zenith distance the effect of parallax is $60' \sin z$, and as this is equal to 45′, $\sin z = 0{\cdot}7500$. Hence $z = 48°\ 36'$.

Measurement of the Moon's Distance

The moon's distance is measured by making use of the same principle utilized by a surveyor who measures a base line and two angles. Having done this, the triangle is easily solved and the lengths of the other sides determined.

Fig. 30 shows the method as applied to finding the distance of the moon M, the base line being O_1O_2 where O_1 and O_2 are two observatories separated by as great a distance as possible, one in each hemisphere.

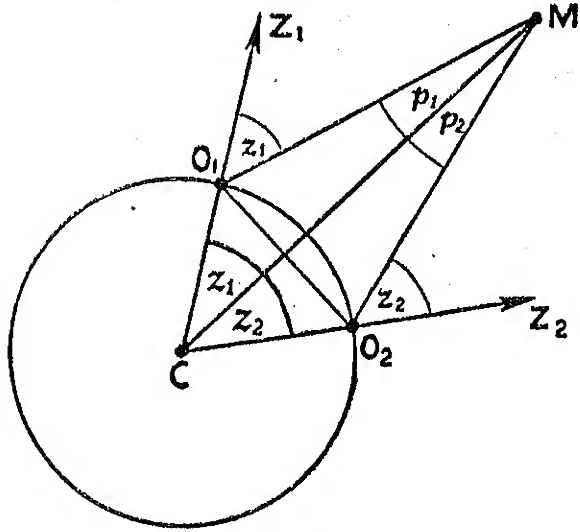

FIG. 30
Measurement of the moon's distance from her parallax at two observatories.

CO_1Z_1 and CO_2Z_2 are the lines drawn from the centre C of the earth to each observatory and these lines pass through the zenith of each observatory. The zenith distances of the moon, Z_1O_1M, Z_2O_2M, at each place are measured, and since the latitudes of the place are known the angle O_1CO_2 is known and hence the equal angles CO_1O_2 and CO_2O_1. (The earth is so nearly spherical that $CO_1 = CO_2$ with sufficient accuracy.)

It is assumed, to simplify the problem, that the two observatories are on the same meridian of longitude so that the moon will transit at the same instant at each observatory. The angle CO_1M is equal to $180° - \widehat{Z_1O_1M}$ and hence is known, and the angle O_2O_1M is equal to $CO_1M - \widehat{CO_1O_2}$. As the angle CO_1O_2 is known the angle O_2O_1M can be found. In the same way the angle O_1O_2M can be found, and as the chord O_1O_2

can be computed, knowing the angle O_1CO_2 and the sides O_1C, O_2C in the triangle O_1CO_2, the triangle O_1O_2M can be solved and the lengths O_1M, O_2M found. When either of these is known the length MC can be computed, and hence the distance of the moon from the centre of the earth—the geocentric distance—is determined.

Observations conducted over a prolonged period show that the moon's distances from the earth vary from about 226,000 to 252,000 miles, the mean distance being a little less than 240,000 miles. (English miles, not nautical miles, are used to express the distances of the heavenly bodies.)

Relation Between the Semi-diameter and the Parallax of a Body

The angle subtended at the earth's centre by the radius of the moon (or any of the other bodies of the solar system) is called its semi-diameter, and there is an important relation between this and the horizontal parallax. In Fig. 31 let M be the centre of the moon and C the centre of the earth, and let r and d be the moon's radius and the distance between her centre and the earth's centre, each being expressed in miles. If S is the angle MCP, or the moon's angular semi-diameter, CP being the tangent to the moon from C, then

$$\sin S = r/d \qquad \ldots \qquad \ldots \qquad \ldots \qquad (41)$$

It has been shown that

$$\sin P = a/d, \text{ or } d = a/\sin P$$

hence

$$\sin S = \frac{r}{a} \sin P \qquad \ldots \qquad \ldots \qquad \ldots \qquad (42)$$

For the same reasons that (38) was expressed more simply in the form (39), we can express (42) in a simple form because S and P are small angles; hence

$$S = P \cdot \frac{r}{a} \qquad \ldots \qquad \ldots \qquad \ldots \qquad (43)$$

Having found the distance d of the moon, (40) enables us to find P and then from (43) we obtain r when S has been measured. The radius of the moon, obtained in this way, has been found to be 1080 miles.

When a relation has once been established between the distance of the moon and her horizontal parallax, by measuring her angular semi-diameter her radius is found, and then, by measuring her angular semi-diameter at any time her parallax is found by (42) and hence her distance. This may seem a little complicated, but an example will show that it is not so difficult as it appears.

Example 4

Suppose the distance of the moon from the earth's centre, determined, by the trigonometrical method already described, is found to be 235,640 miles and the angular semi-diameter at the time is $15' \, 45''.36$. Find the moon's horizontal parallax and also her radius.

By (40) $P = 206{,}265'' \times 3963/235{,}640 = 3469''$ to the nearest second of arc.

By (43) $945''.36 = 3469'' \, r/3963$, from which

$$r = 1080 \text{ miles.}$$

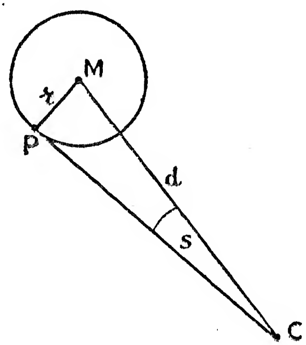

FIG. 31
Showing how to derive a relation between the parallax of a heavenly body and its semi-diameter.

Example 5

Check the consistency of the following data taken from the *N.A.* for 1946, p. 57, assuming that the moon's radius is 1080 miles:

Moon's semi-diameter ..	16′ 18″.31	July 24·0
,, horizontal parallax	59 50·53	July 24·0

By (43) $S = 3590''.53 \times 1080/3963{\cdot}35 = 978''.41$.

The value given in the *N.A.* is $978''.31$. The small discrepancy disappears if we take the value of r/a to be 0·27247 which differs by 3 units in the 5th decimal from the adopted value.

The sun's distance from the earth can be measured in the same way as that of the moon but this method does not provide accurate results owing to the great distance of the sun. Other methods have been adopted, about which more will be said in a later chapter.

The most recent results give the sun's horizontal parallax as $8''\!.79 \pm 0''\!.001$. This is the angle that the earth's equatorial radius subtends at the sun when he is at the mean distance from us, and this distance is about 93,005,000 miles. Naturally such a great distance cannot be found with such accuracy as the distance of a closer body like the moon, and there is a probable error of $\pm$ 9000 miles in this distance—not a very serious matter in dealing with such large figures.

Numerical Example of Computing the Moon's Horizontal Parallax

The actual method for determining the moon's horizontal parallax and from this her distance from the earth will be shown by an example in which ideal conditions will be assumed—that observers in the northern and southern hemisphere are situated on the same meridian. When they are not on the same meridian certain corrections can be applied, but it is unnecessary to burden the reader with these.

Example 6

Observers at two places O_1 and O_2 in latitudes ϕ_1 and ϕ_2, where $\phi_1 = 51°\ 28'\ 38''\!.2$ N., $\phi_2 = 35°\ 56'\ 02''\!.5$ S., make simultaneous observations of a well defined crater supposed to be on the centre of the moon's disc. The observed zenith distances (uncorrected for refraction) are $36°\ 46'\ 58''\!.56$ and $51°\ 59'\ 56''\!.13$ respectively. Find the moon's horizontal parallax and her distance from the earth.

A computing machine has been used in the calculations and the reader should check the results by means of logarithms.

$$\text{By (33)}\ R_1 = 58''\!.2 \tan 36°\ 47' = 43''\!.44$$
$$R_2 = 58{\cdot}2 \tan 52\ \ 00 = 74{\cdot}50$$

(It is sufficiently accurate to take the zenith distance correct to the nearest minute in determining R.)

Hence the corrected zenith distances of the moon's centre at each place are

z_1	..	36°	47′	42″.10
z_2	..	52	01	10·63

In Fig. 30 Z_1 and Z_2 are the true zenith distances corresponding to the apparent zenith distances z_1, z_2, and p_1 and p_2 are the parallaxes at O_1 and O_2 respectively.

In the triangle CO_1M the exterior angle z_1 is equal to the sum of the two interior angles Z_1 and p_1, and a similar relation holds for the triangle CO_2M; hence we have the following relation

$$p_1 = z_1 - Z_1 \qquad p_2 = z_2 - Z_2$$

Now $\zeta_1 + \zeta_2 = \widehat{O_1CO_2} = 87° \, 24' \, 40''.70$. The angle O_1CO_2 measures the difference between the latitudes of the two places, the + sign being used because the places are in different hemispheres. If they had been in the same hemisphere the difference in latitude would have been $15° \, 32' \, 35''.70$, in which case a shorter base line than O_1O_2 would have been available and less accuracy would be attained.

Hence

$$p_1 + p_2 = z_1 - \zeta_1 + z_2 - \zeta_2 = z_1 + z_2 - (\phi_1 + \phi_2).$$

By (39) $p_1 = P \sin z_1$, and $p_2 = P \sin z_2$, hence

$$P\,(\sin 36° \, 47' \, 42''.10 + \sin 52° \, 01' \, 10''.63) = 88° \, 48' \, 52''.73 - 87° \, 24' \, 40''.70.$$

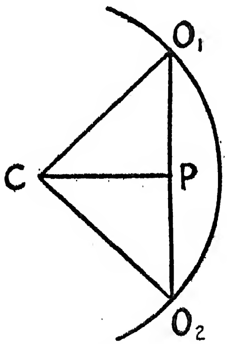

FIG. 32

Finding the length of a straight line connecting two observatories. See text for explanation.

Substituting the values for the sines of each of the angles we have

$$\begin{aligned} P\,(0{\cdot}598{,}954 + 0{\cdot}78822) &= 1° \, 24' \, 12''.03 \text{ or} \\ 1{\cdot}3871774\, P &= 5052''.03 \\ P &= 3641''.96. \end{aligned}$$

From (40) $d = 206{,}265 \times 3963/3641{\cdot}96 = 224{,}447$ miles.

To find the distance from the centre of the earth to the centre of the moon it is necessary to add on the moon's radius, 1080 miles, to the above figures, and the result is 225,527 miles.

The first method described can be used and leads to almost the same results. As a matter of interest it will be shown how the principle explained is applied, and although the work by logarithms may be rather

laborious, some readers will probably wish to check the results which have been attained with a computing machine.

Referring to Fig. 30 and applying the elementary properties of plain triangles, the following results will be obvious:

$\widehat{O_1CO_2}$	87°	24′	40″.70	$\widehat{CO_1O_2} = \widehat{CO_2O_1}$, and hence $2\,\widehat{CO_1O_2} = 180° - \widehat{O_1CO_2}$. Hence
$\widehat{CO_1O_2}$	46	17	39.65	
$\widehat{Z_1O_1M}$	36	47	42.10	
$\widehat{CO_1O_2} + \widehat{ZO_1M_1}$	83	05	21.75	
$\widehat{MO_1O_2}$	96	54	38.25,	
				because $\widehat{MO_1O_2} = 180° - (\widehat{Z_1O_1M} + \widehat{CO_1O_2})$
$\widehat{CO_2O_1}$	46	17	39.65	
$\widehat{Z_2O_2M}$	52	01	10.63	
$\widehat{CO_2O_1} + \widehat{ZO_2M_2}$	98	18	50.28	
$\widehat{MO_2O_1}$	81	41	09.72,	
				because $\widehat{MO_2O_1} = 180° - (\widehat{Z_2O_2M} + \widehat{CO_2O_1})$
$\widehat{MO_1O_2} + \widehat{MO_2O_1}$	178	35	47.97	
$\widehat{O_1MO_0}$	1	24	12.03	

$O_1M/O_1O_2 = \sin \widehat{O_1O_2M}/\sin \widehat{O_1MO_2} = \sin 81°\,41'\,09''.72/\sin 1°\,24'\,12''.03.$

$$\text{Hence } O_1M = O_1O_2 \times 0{\cdot}989{,}491/0{\cdot}024{,}490 = 40{\cdot}4034 \times O_1O_2.$$

The next step is to find the length of O_1O_2. This is easily done as follows.

In the triangle O_1CO_2 (Fig. 32) draw CP perpendicular to O_1O_2. Then $O_1P = O_1C \cos \widehat{CO_1P}$, and $O_2P = O_2C \cos \widehat{CO_2P}$, and hence, since $O_1C = O_2C = 3963$ miles,* and $\widehat{CO_1P} = \widehat{CO_2P}$,

$$O_1O_2 = 2 \times 3963 \cos 46°\ 17'\ 39''.65.$$

Substituting the value of the cosines of the above angle, we have

$$O_1O_2 = 2 \times 3963 \times 0{\cdot}690952 = 5476 \text{ miles.}$$

Hence $O_1M = 40{\cdot}4034 \times 5476 = 221{,}245$ miles.

* This assumption is not, of course, correct, nor is it correct to take the earth's radius as 3963 miles as the earth is not a sphere. The error introduced by the assumptions is not very large, and the main object of the example is to illustrate the method.

We have still to find the geocentric distance of M, and to do so it is necessary to solve the triangle MO_1C, given $MO_1 = 221{,}245$ miles, $O_1C = 3963$ miles, and the angle $\widehat{MO_1C} = 143° \; 12' \; 17''.9$. This is obtained by deducting $\widehat{MO_1Z_1}$, which is $36° \; 47' \; 42''.10$, from $180°$.

In the triangle MO_1C,

$$MC^2 = MO_1^2 + O_1C^2 - 2 \, MO_1 \, . \, O_1C \cos \widehat{MO_1C}.$$

$\cos \widehat{MO_1C} = -\,0{\cdot}800784$, and writing the above in the form,

$$MC^2 = MO_1^2 \left(1 + \frac{O_1C^2}{MO_1^2} + 2\frac{O_1C}{MO_1} \times 0{\cdot}800784\right)$$

we obtain, on substituting the value of O_1C, that is, 3963 miles, and of O_1C/MO_1, which is $0{\cdot}017912$,

$$MC^2 = MO_1^2 \, (1 + 0{\cdot}017{,}912^2 + 0{\cdot}028687).$$
$$\text{Hence } MC^2 = 221{,}245^2 \times 1{\cdot}029008.$$

$$MC = 221{,}245 \sqrt{1{\cdot}029{,}008} = 221{,}245 \times 1{\cdot}01{,}440 = 224{,}431 \text{ miles.}$$

This is the distance to the crater on the surface of the moon, and adding 1080 miles to this, the geocentric distance of the centre of the moon is 225,511 miles. The value found by the other method was 225,527 miles, and the difference of 16 miles is probably due to an accumulation of small errors. Both methods have been shown to let the reader see that the method which makes use of the parallaxes is very much shorter and should always be used.

Example 7

What are the moon's semi-diameters as seen from O_1 and C?

$\sin S = r/d$, or, with sufficient accuracy, $S = 206{,}265'' \, r/d$. In the first case $d = 222{,}325$ and hence $r/d = 0{\cdot}0048{,}577$. In the second case $d = 225{,}527$ and $r/d = 0{\cdot}0047{,}888$. Hence the moon's semi-diameter at O is $1000''.20 = 16' \; 40''.20$, and at C it is $987''.76 = 16' \; 27''.76$.

If S_0 is the moon's semi-diameter at any place O on the surface of the earth and S is the geocentric semi-diameter, the distances from the moon being d_0 and d in each case, then

$$S_0 = S \, . \, d/d_0.$$

Thus in the above example, if d is 225,527 and d_0 is 222,325,

$$S_0 = 987''.76 \times 225{,}527/222{,}325 = 987''.76 \times 1{\cdot}0144 = 1000''.20.$$

The moon's semi-diameters vary between $16' \; 45''$ and $14' \; 42''$ approximately, these variations occurring owing to the elliptic motion of the

moon round the earth. The moon does not actually move round the earth's centre but this will be dealt with in a later chapter.

The Sun's Horizontal Parallax

The *N.A.* gives the sun's horizontal parallax for every 10 days in the year, and this is sufficient because his parallax does not change quickly as in the case of the moon. On p. 46 of the *N.A.* for 1946 the sun's horizontal parallax for the beginning of the year is given as $8''.95$ and for June 29 it is $8''.66$. The sun is nearest to and at greatest distance from the earth about these periods, and its distance from the earth in each case can be found by (40). In the first case this is 91,449,000 miles and in the second case it is 94,561,000 miles. It should be noted that only three figures are given in the parallax, and hence there is some uncertainty in the fourth figure in the computed distances, while the figures from the fifth to the eighth are quite meaningless, and zeros or other figures could be used in place of those given (see p. 97 on probable error of sun's distance). Thus, instead of 91,449,000, we could take the distance as 91,449,479 and it would probably be as accurate as that given above. In the figures for the sun's distance an error of 5000 miles is like an error of an inch in 500 yards.

Stellar Parallaxes

It has been shown that the earth's radius subtends such a small angle at the distance of the nearest star that it is impossible to detect this angle. A much larger base-line is necessary, and this is provided by the diameter of the earth's orbit in its motion round the sun. As this diameter is about 186 million miles one might imagine that it would be a very satisfactory base line, but unfortunately it is much too small except for the comparatively close stars. The method of determining the parallaxes of stars—sometimes called *annual parallaxes* because they depend upon the earth's annual motion round the sun—will be better understood by referring to Fig. 33.

Let S be a star at a distance d (to be determined) from the sun, and let a be the radius of the earth's orbit assumed to be circular to simplify the problem. Just as a background of stars was necessary in finding the moon's distance from the earth, so a background of stars is necessary in finding the distance of a star from the sun. In the latter case any star will not do because it may be too close, so it is necessary to select a background of faint stars which may be presumed, from their faintness, to be very far away from the earth—much further than the star whose parallax we wish to find.

Let E and E' be the positions of the earth in its orbit at two periods separated by an interval of six months, so that EE' is a base line of about

186 million miles. Suppose it is required to measure the parallax of a star S whose direction from the sun is practically at right angles to EE'. This is the simplest case, but when the star lies in a position which does not comply with this condition corrections can be applied. Now imagine that there is a faint star S' which lies in the plane SEE' and which is very far distant from the sun—so far that the lines ES' and $E'S'$ can be considered parallel. A similar assumption is made when observations of the pole star are made from different places on the earth's surface to find the latitudes of the places, but in this case the distance between the observers is only a minute fraction of the length EE'. The angles $SE'S'$ and SES' are measured carefully and from these the angle ESE' is easily obtained as follows.

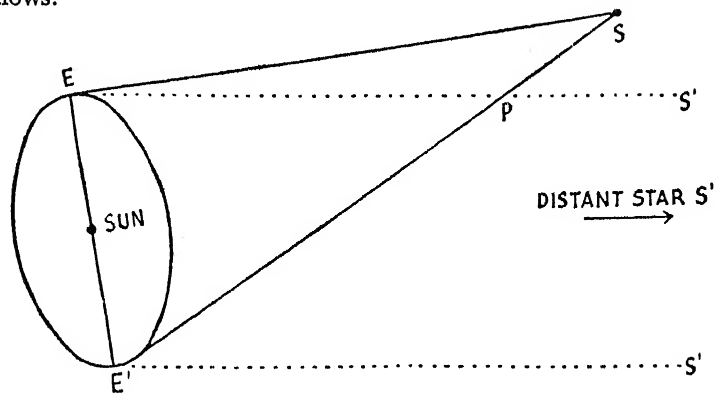

FIG. 33
Showing how the parallax of a star is found.

The angle $SE'S'$ is equal to the angle EPE' because ES' and $E'S'$ are parallel, and the angle EPE' is equal to the sum of the angles ESP and SEP, or the angle $SE'S'$ is equal to the sum of the angles ESE' and SEP. Hence $\widehat{SE'S'} - \widehat{SEP} = \widehat{ESE'}$. The angle ESE' is not the parallax; the parallax is the angle subtended by the *radius* of the earth's orbit, just as the parallax of the sun, moon or a planet is the angle subtended by the radius of the earth at the body in each case.

In the simple case under consideration, if we imagine that the parallax of S is found to be $0''.25$, then, because the line drawn from S to the sun is perpendicular to EE', the distance of the star is $93{,}005{,}000 \times 206{,}265/0{\cdot}25$, or 77×10^{12} miles approximately.

The parallax of a star is the maximum angle subtended at the star by the line joining the earth and the sun, and in cases where the line sun-star is not at right angles to the line sun-earth, although the angle subtended at the star is obviously less than in the case considered, reductions are always made so that the parallax refers to the maximum angle.

It has been assumed that the faint star is at an infinite distance, but

as this assumption is not quite valid the results obtained are only the relative parallax, or the parallax with reference to some other star. If the distance of the faint star can be determined (and other methods besides the trigonometrical one just described are used for finding stellar distances) the absolute parallax of the brighter star is obtained by adding its relative parallax to the parallax of the faint star. The corrections to apply in such cases are usually very small.

It is not convenient to use miles or even the astronomical unit to express stellar distances, and other units have been adopted. One very convenient unit is the light-year, which is the distance through which light would travel in a year. As light travels with a speed of 186,271 miles a second, this is equivalent to $5{\cdot}88 \times 10^{12}$ miles a year. Another unit is the *parsec*, which is the distance corresponding to a parallax of one second of arc. Since $P'' = 206{,}265''\, a/d$, if P is $1''$ $d = 206{,}265\, a$. Expressing a as one astronomical unit, it follows that a parsec is 206,265 astronomical units. But an astronomical unit is 93,005,000 miles, and hence a parsec is $19{\cdot}183 \times 10^{12}$ miles. Since a light-year is $5{\cdot}88 \times 10^{12}$ miles, it follows that a parsec is $19{\cdot}183 \times 10^{12}/(5{\cdot}88 \times 10^{12}) = 3{\cdot}26$ light-years. The relations between the various units are shown below.

One astronomical unit ..	93,005,000 miles
One light-year	$5{\cdot}88 \times 10^{12}$ miles = 63,222 astronomical units
One parsec	$19{\cdot}183 \times 10^{12}$ miles = $3{\cdot}26$ light-years .. .. (44)

If a star has a parallax of p'' its distance is $1/p$ parsecs.

Example 8

What is the distance of Sirius in miles, astronomical units, parsecs and light-years, if its parallax is $0''{\cdot}371$?

$d = 206{,}265 \times 93{,}005{,}000/0{\cdot}371 = 51{\cdot}7 \times 10^{12}$ miles.
$d = 206{,}265 \times 1/0{\cdot}371 = 555{,}970$ astronomical units.
$d = 1/0{\cdot}371 = 2{\cdot}69$ parsecs.
$d = 2{\cdot}69 \times 3{\cdot}26 = 8{\cdot}8$ light-years.

Example 9

The nearest star to the earth is Proxima Centauri, whose parallax is $0''{\cdot}783$. What is its distance in light-years?

Its distance in parsecs is $1/0{\cdot}783 = 1{\cdot}28$. Hence its distance in light-years is $1{\cdot}28 \times 3{\cdot}26 = 4{\cdot}18$.

Problems

1. The observed zenith distance of the sun, uncorrected for refraction, is $25°$, and his horizontal parallax is $8''.76$. Find his true zenith distance.

2. About the middle of October the sun's horizontal parallax is $8''.83$. Find his distance from the earth at that time.

3. The *N.A.* gives the moon's horizontal parallax on 1946 December 9.0 as $61' \; 28''.18$ and her semi-diameter as $16' \; 44''.91$. Find her distance from the earth and also her diameter in miles.

4. If the moon's altitude above the horizon, corrected for refraction, is $32° \; 16' \; 17''.8$, find her true zenith distance if her horizontal parallax is $53' \; 58''.90$.

5. On 1946 May 17, the horizontal parallax of Venus is $5''.96$, and her semi-diameter is $5''.70$. Find the distance of Venus from the earth and also her diameter in miles.

6. On 1946 August 1, the polar semi-diameter of Jupiter is $16''.23$. Assuming that the polar diameter of Jupiter is 82,800 miles, find his distance from the earth on the above date.

7. The moon's maximum and minimum horizontal parallaxes are about $60'.3$ and $54'.0$. Find the maximum and minimum distances of the moon from the earth.

8. Find the maximum and minimum values of the moon's angular semi-diameter from the data in 7.

9. What must be the parallax of a star if its light has been travelling since the Battle of Waterloo in 1815 and reaches the earth in 1946?

10. The sun's horizontal parallax on December 31 is $8''.95$ and his semi-diameter subtends an angle of $16' \; 17''.51$. Find the distance of the sun from the earth at the time and also his diameter.

11. A spot is observed on the sun near the centre of his disc about May and subtends an angle of $3''$ at the earth. What is the diameter of the spot in miles?

12. The moon's horizontal parallax is $59' \; 12''.35$ and the angle subtended at a place on the surface of the earth by the crater Triesnecker near the centre of the moon's disc is $125''$. What is the diameter of the crater?

13. What assumption has been made in 12 and why is it justified?

It should be noticed that in some of the above examples only three significant figures are available for the parallax and semi-diameter. In such cases the fourth significant figure in the computations cannot be exact and the remaining figures are meaningless although they appear in the answers. The *N.A.* supplies the values of the distances of the sun and the planets from the earth for each day in terms of an astronomical unit.

Chapter VII

ABERRATION, PRECESSION AND NUTATION

The phenomenon of aberration is due to the fact that the velocity of light is finite—186,271 miles, or, as it is often expressed, 3×10^{10} cm.—per second. In 1725 Bradley started a series of observations of the star γ Draconis with the object of measuring its parallax. He noticed certain discrepancies which were inexplicable at first, but in 1728 he was able to explain these by the phenomenon of aberration, a description of which follows.

Illustration of Aberration

A familiar illustration of aberration is usually given in text-books and affords quite a simple explanation. The illustration refers to the method adopted for protection against drops of rain which, we may suppose, is falling vertically, while someone who is carrying an umbrella is walking through the rain and holding the umbrella over his head.

In the first instance, if the person is standing still he holds the umbrella straight over his head, but if he starts walking he finds that it is necessary to hold it in a slanting position and inclined in the direction of his motion. In addition, the faster he walks the greater the slope of the umbrella. Although we assume that the rain is falling vertically, the *apparent* direction in which it is falling when the man is walking is not vertical but slightly inclined to the vertical. It must be remembered that another person who was standing still and looking on would see the rain falling vertically, but the one who is walking sees it falling at a slope with reference to himself. If he stands still the rain appears to fall vertically.

Determination of the Constant of Aberration

The principle involved is that of the parallelogram of velocities. To explain how the position of a star is displaced owing to the earth's orbital velocity, let O be an observer on the earth and $A'OA$ the direction of the earth's motion at any instant. Let OS be the true direction of a star S (see Fig. 34). On the tangent $A'OA$ to the earth's orbit take OA to represent the earth's velocity in magnitude and OK, in SO produced, the velocity of light on the same scale.

The relative motion of the light with reference to the earth will not be altered if a common velocity is given to each, and it will be assumed that this common velocity is OA', which is equal and opposite to OA. The earth will be brought to rest and the velocity of the light from the star will be represented by OW, the diagonal of the parallelogram $A'OKW$. If WO is produced the direction in which the star will be seen is OS' and the angle SOS' is called the *aberration* of the star.

In the first instance, suppose that the star is in the pole of the ecliptic so that its light is moving at right angles to the direction of the earth's motion. The parallelogram could have been constructed with OA and

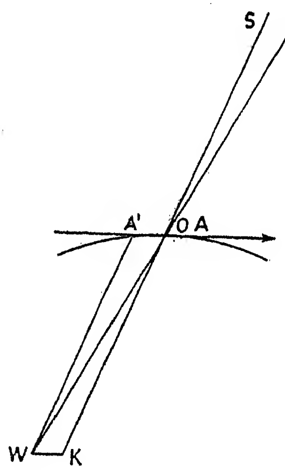

FIG. 34
Explanation of aberration.

OS as sides, the diagonal OS' representing the direction in which the star is seen, and hence Fig. 35 can be used to determine the effect of aberration, the light from the star reaching the earth at right angles to its orbital motion. A telescope would not be pointed in the direction OS but in the direction OS' to see the star.

Let v be the earth's orbital velocity and c the velocity of light. In the triangle SOS', if $\widehat{S'SO}$ is supposed to be a right angle, we have the relation

$$\tan \widehat{SOS'} = SS'/SO = v/c \qquad \ldots \qquad \ldots \qquad \ldots \qquad (45)$$

When the earth is at its mean distance from the sun, $v = 18{\cdot}49$ miles per second, and hence $\tan \widehat{SOS'} = 18{\cdot}49/186{,}271 = 0{\cdot}00009926$. Hence $\widehat{SOS'} = 20''.47$.

In this particular case the angle SOS', denoted by α, is called the *constant of aberration.* It should be noticed that $\tan \alpha = 0{\cdot}00009926$; and because α is a very small angle, $\tan \alpha = \sin \alpha =$ the radian measure of α.

When the direction of the light from the star is not at right angles to the direction of the earth's orbital motion, we have, from Fig. 35,

$$\sin \widehat{SOS'} = \frac{SS'}{SO} \sin \widehat{SS'O} = \tan \alpha \sin \widehat{SS'O} \qquad .. \qquad (46)$$

Since $\widehat{SOS'}$ is smaller than α, which we have shown to be so small that $\sin \alpha = \tan \alpha =$ radian measure of α, we can express both α and $\widehat{SOS'}$ in radian measure, and obtain the simple relation,

$$\text{abberration} = \text{constant of aberration} \times \sin \widehat{SS'O} \qquad .. \qquad (47)$$

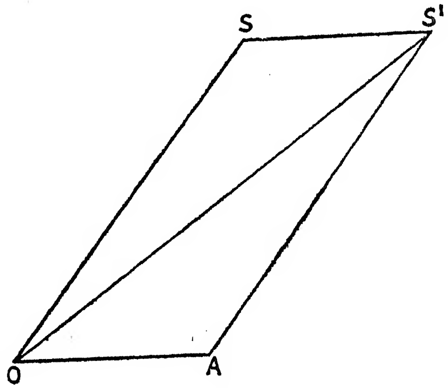

FIG. 35
Determination of the constant of aberration.

The constant of aberration can be defined as the apparent displacement of a star when the earth is moving with average speed at right angles to the star's direction.

The angle $SS'O$ is equal to the angle $S'OA$, which is practically equal to the angle SOA, because $\widehat{SOS'}$ is very small. The angle SOA between the lines drawn from O to the star and in the direction of the earth's motion is known as the earth's way, and hence we have the relation

$$\text{aberration} = \text{constant of aberration} \times \sin \text{earth's way} \qquad (48)$$

We have described the effects of aberration in displacing the position of a star and have shown that the maximum effect of this displacement is $20''.47$. If the star is directly in front of or behind the direction of the

earth's motion, the earth's way is 0° or 180°, and hence the aberration is zero. In all other cases corrections in the right ascension and declination of the star must be made, and the *Nautical Almanac* provides certain constants which can be used in the computations. These will be dealt with later when corrections for other phenomena are considered.

Diurnal Aberration

The aberration with which we have just dealt is due to the earth's orbital velocity, but there is another kind of aberration which is due to the earth's daily rotation. Suppose an observer is at the equator where the velocity of the earth, due to its axial rotation, is about 0·288 mile per second, which is $0{\cdot}288/18{\cdot}49 = 0{\cdot}01557$ times the earth's average orbital velocity, then the aberration effect will be

$$\text{Diurnal aberration} = 0{\cdot}01557 \times 20''.47 = 0''.32 \qquad \ldots \quad (49)$$

At a latitude ϕ the effect will be $0''.32 \cos \phi$, or, if κ is the diurnal aberration at any place with latitude ϕ,

$$\kappa = 0''.32 \cos \phi \qquad \ldots \quad \ldots \quad \ldots \quad (50)$$

The effect of the diurnal aberration is a maximum at the equator and vanishes at either pole.

The effect of diurnal aberration is so small that it can generally be neglected, but if a star is near the pole it should be taken into consideration. When a star is on the meridian its right ascension is increased by the diurnal aberration by an amount

$$0''.32 \cos \phi \sec \delta = 0^{s}.0213 \cos \phi \sec \delta.$$

If $\phi = 0°$, that is, if the observer is in equatorial regions, and if the star is near the equator so that $\delta = 0°$, or $\sec \delta = 1$, the time of transit of the star will be delayed by 0·0213 second of time, which would be difficult to observe. On the other hand, if the declination of the star is 85°, then since $\sec 85° = 11.47$, the retardation in this case would be $11.47 \times 0{\cdot}0213 = 0{\cdot}24$ second, which would be appreciable. This applies to the case of an observer at the equator, but at the latitude of Greenwich, where $\cos \phi = 0{\cdot}6228$, the retardation would be only 0·15 second.

Planetary Aberration

Just as the direction of a star is affected by aberration which is due to the motion of the earth, so the apparent direction of a planet or other body in the solar system is affected by the motion of the earth, and in

addition, by the motion of the planet or other body in the solar system. It will be recalled that a star is so far away from the earth that its motion can be ignored unless it is considered over a long period, but as the bodies in the solar system are comparatively close to us, their motions must be taken into consideration.

To show the effects of aberration in the case of a planet we shall take the particular case of Venus, but all the planets can be dealt with in a similar manner.

In Fig. 36 E and E' are two positions of the earth in its orbit represented by the outer circle. The inner circle represents the orbit of Venus, V and V' being two positions of the planet. Suppose that the distance from V to E is 30,000,000 miles. Then since light travels at a speed of 186,271 miles per second, the light from Venus will require 161 seconds to reach the earth. Let EE' be the distance that the earth moves

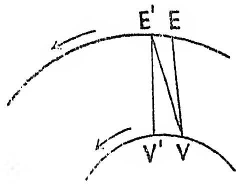

FIG. 36
Explanation of planetary aberration.

over in 161 seconds and also let VV' be the distance that Venus travels in her orbit in the same time. The light which leaves Venus when she is at V reaches the earth when it is at E', and hence the direction of the actual motion of the light is VE', so we have the relation

$$EE'/VE' = \text{orbital velocity of earth/velocity of light.}$$

From Fig. 36 it is obvious that VE represents the direction of relative velocity of the light with respect to the earth, and hence when the earth is at E' Venus is seen in a direction parallel to EV. But V was the position of Venus 161 seconds previously, and hence the apparent direction of Venus is just what its real direction was 161 seconds previously. It should be noticed that the above correction for relative motion automatically includes stellar aberration.

The same argument applies to the sun and other bodies in the solar system, provided the path of the earth can be taken as a straight line in the interval. As the light-time for bodies in the solar system is always relatively short, the above condition holds with sufficient accuracy.

The mean distance of the sun from the earth is 93,005,000 miles, and light travels this distance in 499·3 seconds, or say 500 seconds. Hence,

expressing the distance of an object from the earth as ρ astronomical units, its true position can be determined by finding what it was 500ρ seconds previously.

As an illustration of the above principle, take the following example.

Example 1

On 1946 August 16, Jupiter's distance from the earth was 5·5876 astronomical units. What is the relation between his actual and apparent co-ordinates?

$$5{\cdot}5876 \times 500 = 2938 \text{ seconds.}$$

Hence Jupiter's apparent co-ordinates are his actual co-ordinates $48^{\mathrm{m}}\ 58^{\mathrm{s}}$ previously.

Precession

The precession of the equinoxes is caused by the pull of the sun and the moon (the moon especially, owing to the fact that it is so close to the earth, though its mass is only $1/(27 \times 10^6)$ that of the sun) on the equatorial bulge of the earth. Because the protuberance at the equator is slightly nearer the sun and moon than are the other portions of the earth, the attraction there is greater, and hence the tendency of the pull of the sun and moon is to make the equator coincide with the ecliptic. As the earth is rotating there is a gyroscopic effect which can be illustrated very easily by spinning a top and suspending it as shown in Fig. 37.

It will be observed that the force of gravity tends to make the axis of the top coincide with the perpendicular to the horizon. If there were no spin and the top were suspended as shown, this coincidence would take place, but other forces are set up when spinning occurs and the whole apparatus precesses round the vertical in a direction opposite to that of the spin of the top. If we imagine that the axis of the spinning top represents the earth's axis and that the horizon represents the ecliptic, we have a good illustration of the phenomenon of precession. The horizon can be regarded as fixed while the axis of the top moves round it and maintains a constant angle (for a very short period) with the horizon. The effect of precession is that the earth's axis performs a slow conical movement round a line joining the poles of the ecliptic, a complete precession taking place in 25,800 years (see Fig. 38).

Observations of the positions of the stars by Hipparchus about 125 B.C. led to the conclusion that while the ecliptic is practically a fixed great circle on the celestial sphere with reference to the background of stars, yet the equator E alters so that the first point of Aries is carried backwards

on the ecliptic. He did not know the cause of the phenomenon, but was able to measure with a fair degree of accuracy the effect of precession. This effect is to make ♈ move backwards along the ecliptic at the rate of 50″.2 a year, and hence the longitudes of the stars increase by this amount

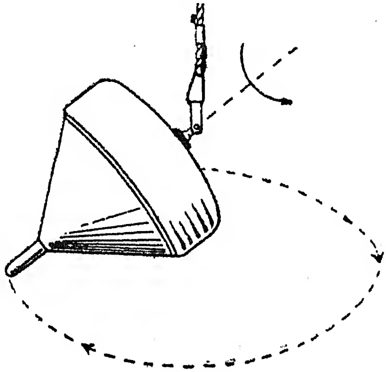

Fig. 37
A spinning top suspended by a cord used to explain precession.

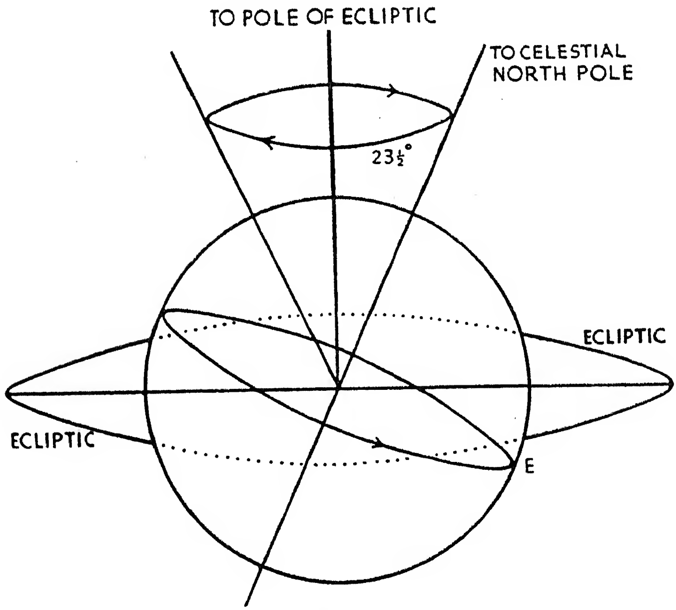

Fig. 38
Showing the phenomenon of the precession of the equinoxes.

each year while their latitudes remain unchanged. The change in longitude implies changes in right ascension and declination of the stars, and hence it is necessary when we describe the equatorial co-ordinates of a star to specify the year for which the co-ordinates are reckoned. Thus, if we say that the right ascension and declination of a star are 3^h and $60°$ respectively, this does not supply very accurate information unless we specify the time for which the reckoning is made.

It is usual to give the positions of the stars for the equator and equinox for the beginning of the year, and in these circumstances we describe these as the mean equator and mean equinox for the beginning of the year, written in the form 1946·0 for the year 1946, and so on. This method is not always adopted and varies with the star catalogues, and in addition, the year 1950.0 has been adopted for certain purposes in describing the positions of heavenly bodies.

Computation of Precessional Effects

The mean co-ordinates as thus defined can be found for any other year up to a period of about 40 years with sufficient accuracy by the following formulae, α and δ denoting the mean co-ordinates for the year t, and α_1 and δ_1 the mean co-ordinates for the year $t + n$.

$$\begin{aligned} \alpha_1 - \alpha &= n\,(3^s\!.073 + 1^s\!.336 \sin \alpha \tan \delta) \\ \delta_1 - \delta &= n\,(20''\!.04 \cos \alpha) \qquad \ldots \qquad (51) \end{aligned}$$

If we wish to find the co-ordinates for an earlier year—say to transfer the co-ordinates from 1946.0 to 1940.0—we make n negative in the above expressions.

Nutation

The moon's orbit is inclined at more than $5°$ to the ecliptic, and intersects the ecliptic in two points known as the nodes. These nodes have a motion round the ecliptic, completing a revolution in less than 19 years, and during this time the inclination of the moon's orbit to the equator varies between $23\frac{1}{2}° \pm 5°$, that is, between $18\frac{1}{2}°$ and $28\frac{1}{2}°$. Her effect on the earth's equatorial regions varies also owing to the different inclinations, and hence precession does not proceed at a uniform rate. The result is that the curve described by the axis of the earth is not exactly a circle but fluctuates slightly, the pole "nodding", for which reason this phenomenon is called nutation (from the Latin *nutare*, to nod).

Although (51) gives accurate co-ordinates of the stars, provided the interval is not too long, it takes no account of nutation effects. These

must be taken into consideration in all cases where accuracy is required, and formulae for computing nutation, etc., will be given later when we come to deal with certain constants given in the *N.A.*

Proper Motion

The subject of proper motion will be considered more fully later. Meanwhile it will be sufficient to say that all the stars are in motion and the annual change in heliocentric direction on the celestial sphere, due to a star's motion through space, is called its *proper motion*. The proper motions of a number of stars have been calculated with considerable accuracy, and these are given in right ascension and declination to enable the star's co-ordinates to be recorded with precision.

The Tropical Year and the Sidereal Year

Up to the present we have defined the year as the interval required by the sun to complete a circuit of the ecliptic, and this period is called the *sidereal year*. From what has just been said about precession and nutation it is evident that a sidereal year does not correspond to the interval between two successive passages of the sun through ♈ because this point has a backward movement of $50.''2$ yearly on the ecliptic. Hence the sun will reach ♈ sooner on his annual motion amongst the stars than he reaches a defined point with reference to the stars. The interval between successive passages through ♈ is known as the tropical year, the mean value of which is 365.2422 mean solar days. The relation between the two kinds of year can be found as follows:

In a sidereal year the sun moves through 360° and in a tropical year he moves through $50.''2$ less than 360°, that is, through $1{,}296{,}000'' - 50.''2 = 1{,}295{,}949.''8$. Hence,

$$1 \text{ tropical year}/1{,}295{,}949{\cdot}8 = 1 \text{ sidereal year}/1{,}296{,}000,$$

from which we find that a sidereal year is 1·00003935 tropical years. The length of a tropical year is 365·2422 mean solar days, and hence a sidereal year is 365·2564 mean solar days.

Apparent, Mean and True Places of a Star

It has been shown that there is a movement of the equator and equinox owing to precession and nutation and that it is necessary to define the time for which the co-ordinates of a star are given, as otherwise there would not be a common basis from which astronomers could

work and make their calculations. In addition to precession and nutation, there are other effects which must be taken into consideration and corrections applied for each one of them. The corrections can be included under five heads as follows: (1) Precession; (2) Nutation; (3) Aberration; (4) Annual parallax; (5) The proper motion of the star. Reference has already been made to all of these, and the method for making the necessary corrections will be shown in the example at the end of the chapter.

The *apparent position* of a celestial body is its position on the celestial sphere, as it would be seen if the observer were at the earth's centre. It is referred to the true equator and true equinox at *the instant of observation.*

As the geocentric parallax of a star is negligible, the apparent place of a star is simply its observed position, corrections for refraction being applied. The geocentric parallax of bodies in the solar system cannot be ignored, and hence the apparent position of a planet or other body in the solar system is its observed position on the celestial sphere, corrections for both refraction and parallax having been applied. The co-ordinates are referred to the true equator and true equinox at the instant of observation.

The *true place* of a star is its position as it would be seen by an observer if he could be transferred to the sun. The co-ordinates are referred to the true equator and true equinox at the instant of observation.

If the corrections due to aberration and the annual parallax are applied to the true place of a star the result is its apparent place.

The *mean place* of a star is its position as seen from the sun, but it is referred to the mean equator and mean equinox at the beginning of the year.

If observations of a star are made at different times of the year it is possible to compare them only when they are reduced to some agreed equator and equinox. The equator and equinox for the beginning of the year are used for this purpose, and if the observations are made over a series of years it is necessary to make the reductions from the mean position for one year to the mean position for the beginning of another year.

Independent Day Numbers

The reductions are facilitated by the use of the Besselian Day Numbers and also by the Independent Day Numbers which are given for each day of the year in the *Nautical Almanac.* If the computations were carried out without these the process would be involved and tedious, but, as will be shown, it is a simple matter to make the reductions by using these numbers. If the Besselian Day Numbers are used certain constants for

the particular star must be computed, but if the Independent Day Numbers are used the computation of these constants is unnecessary. We shall therefore illustrate the process of reduction by using the Independent Day Numbers.

Example 2

The mean place for α Orionis for 1946 is $\alpha = 5^h\ 52^m\ 14^s.831$, $\delta = +\ 7^\circ\ 23'\ 55''.3$. Find its apparent place on 1946 December 6.

The formulae for making the corrections are as follows, α and δ being the mean right ascension and declination, α_1 and δ_1 the apparent right ascension and declination on December 6, and μ and μ' the proper motion in right ascension and declination. The constants used in the two equations are given on p. 276 of the *N.A.*

$$\alpha_1 - \alpha = f + g \sin (G + \alpha) \tan \delta + h \sin (H + \alpha) \sec \delta + \mu\tau$$
$$\delta_1 - \delta = g \cos (G + \alpha) + h \cos (H + \alpha) \sin \delta + i \cos \delta + \mu'\tau$$

From these constants we find as follows:

$G + \alpha = 5^h\ 04^m.2$		$H + \alpha = 6^h\ 53^m.9$	
$\log g$	$1\cdot0832$	$\log h$	$1\cdot3083$
$\log \sin (G + \alpha)$	$\bar{1}\cdot9869$	$\log \sin (H + \alpha)$	$\bar{1}\cdot9878$
$\log \tan \delta$	$\bar{1}\cdot1135$	$\log \sec \delta$	$0\cdot0036$
sum	$0\cdot1836$	sum	$1\cdot2997$
antilog	$1''.526$	antilog	$19''.950$

The sum of the second and third terms in the right-hand side of the above expression $= 21''.476$. This must be expressed in seconds of time by dividing it by 15. The result is $1^s.432$. Hence

$$\alpha_1 - \alpha = 1^s.814 + 1^s.432 = 3^s.246.$$

og g	$1\cdot0832$	$\log h$	$1\cdot3083$	$\log i$	$0\cdot3700$
$\log \cos (G + \alpha)$	$\bar{1}\cdot3837$	$\log \cos (H + \alpha)$	$\bar{1}_n\cdot3682$	$\log \cos \delta$	$\bar{1}\cdot9964$
sum	$0\cdot4669$	$\log \sin \delta$	$\bar{1}\cdot1099$	sum	$0\cdot3664$
antilog	$2''.930$	sum	$\bar{1}\cdot7864$	antilog	$2''.325$
		antilog	$-\ 0''.6115$		

$$\delta_1 - \delta = 2''.930 - 0''.611 + 2''.325 = 4''.64.$$

The proper motion has been ignored as it is very small. Its annual value is $\mu = +\ 0^s.0019$ in R.A., and $\mu' = +\ 0''.010$. On December 6 the fraction of the year τ is $0\cdot9283$ (see p. 277 of the *N.A.*), and hence $\mu\tau = +\ 0^s.0018$, $\mu'\tau = +\ 0''.009$. If these are applied to the above results, $\alpha - \alpha = 3^s.248$, $\delta - \delta = 4''.65$. Hence, the mean and apparent places for α Orionis are as follows:

	R.A.			*December*		
Mean place ..	5^h	52^m	$14^s.83$	$+7°$	$23'$	$55''.30$
Apply corrections			$+3.25$			$+4.65$
Apparent place ..	5	52	18·08	$+7$	23	59·95

Four-figure tables are sufficient for the computations, and $G + \alpha$ and $H + \alpha$ can be taken to the nearest minute of arc.

Problems

1. The mean place of α Persei for the year 1946 is as follows: R.A. $3^h\ 20^m\ 27^s.31$, Dec. $+ 49°\ 40'\ 14''.7$. What is its mean position for the year 1940?

2. On 1946 March 1, the apparent right ascension and declination of Saturn are given as follows: $\alpha = 7^h\ 19^m\ 16^s.23$; variation in 1 day $-8^s.75$; $\delta = +22°\ 11'\ 50''.1$; variation in 1 day $+ 22''.8$. The distance of Saturn from the earth at the time is 8·42073 astronomical units. What are the actual co-ordinates of Saturn at the time?

3. Check the following results taken from the *N.A.* for 1946. The mean place of η Centauri for 1946 is $\alpha = 14^h\ 32^m\ 04^s.07$, $\delta = -41°\ 55'\ 18''.4$. Its apparent place for November $6^d.5$ is $\alpha = 14^h\ 32^m\ 04^s.08$, $\delta = -41°\ 55'\ 21''.8$. Use the Independent Day Numbers in the computation. (For proper motions see *N.A.*, pp. 278–83.)

Chapter VIII

THE LAW OF GRAVITATION AND THE MOTIONS OF THE HEAVENLY BODIES

For a long time the motions of the planets were believed to take place in circles. Aristotle taught that the circle was the "perfect figure", and owing to his dominating influence astronomers even as recent as the sixteenth century attempted to reconcile the observed positions of the planets with circular motion. Tycho Brahé (1546–1601) made very accurate observations of the positions of Mars, and the discrepancies between theory and observation were cleared up by Kepler (1571–1630), who abandoned the idea of circular motion and adopted the view that the planets moved in ellipses, the sun being in one of the foci of the ellipse. A short description of the ellipse follows.

The Ellipse

It has been shown in Chapter IV how an ellipse can be traced out on a sheet of paper. If the reader carries out this experiment and varies the distances between the pins, he will be able to trace out a number of ellipses of various shapes—some very elongated and some nearly circular, with intermediate types. Fig. 39 shows an ellipse which resembles the orbits of a few of the minor planets; the orbits of the planets are much more like circles than Fig. 39, and if these orbits were reduced to the scale used in drawing this curve it would be very difficult to distinguish them from circles.

The two points S and S', corresponding to the two pins used in drawing the figure, are the foci of the ellipse, and the line passing through S, S', and terminated by the curve at A and B, is known as the major axis, a. The middle point O of AB is the centre of the ellipse, and the line CD drawn through O perpendicular to AB and, bounded by the curve, is the minor axis, b. The perpendicular to AB through S or S', terminated by the ellipse, is the parameter, p. The ratio SO (or $S'O$) to OA is known as the eccentricity, e, and the greater the eccentricity the more oval is the ellipse. The eccentricity lies between 0 and 1, and if it is exactly 0 the ellipse becomes a circle as can be easily verified by making the pins approach closer and closer and noticing that the curve becomes more circular each time the pins approach each other. When they finally

coincide it will be found that the method for drawing the curve, while still applying, will now trace a circle. When the eccentricity is exactly 1 the figure becomes a parabola, and in this case one of the foci is at an infinite distance. A figure rather like a parabola can be drawn by placing the pins far apart, in which circumstances the portion of the curve near

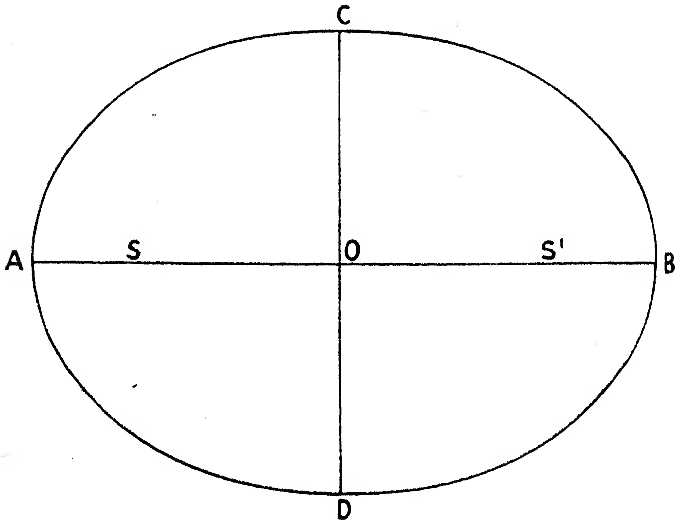

FIG. 39
An ellipse, showing the foci and major axis.

either pin resembles a parabola. A parabola, unlike an ellipse, is an open curve, not closing in again on itself. The hyperbola, in which the eccentricity exceeds 1, is also an open curve.

Generally speaking, most of the considerations regarding the motions of the heavenly bodies will be restricted to motion in an ellipse, in which curve move by far the great majority of celestial bodies, including all the planets and asteroids. Some properties of the ellipse will be dealt with at a later stage; meanwhile the three laws of planetary motion formulated by Kepler will be considered and brief explanations given of each of these.

Kepler's First Law

The orbit of a planet is an ellipse with the sun situated in a focus.

In Fig. 40 S is the sun in one of the foci of an ellipse and P_1, P_2, P_3, P_4 represent various positions of a planet in its revolution round the sun. Kepler's law was applied in the first instance to the orbit of Mars, but it applies to all the planets and also to the satellites; in the latter case the planet to which the satellite or satellites are attached and round which

they revolve as the planets revolve round the sun is the focus of the ellipses described. It has been shown that the sun appears to describe an orbit round the earth—a hypothesis which is often useful for simplifying certain computational problems—though of course it is the earth which describes the orbit relative to the sun which is in one of the foci of the ellipse described. The eccentricity of the earth's orbit is small—about 1/60 — and hence this orbit does not differ very much from a circle.

The points A and P which are at the greatest and least distances from S are called aphelion and perihelion, respectively. The angle θ which a line from S to any point on the ellipse makes with SP is called

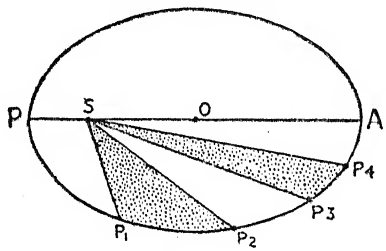

FIG. 40
Explanation of Kepler's first two laws of planetary motion.

the true anomaly, and lines such as SP_1, SP_2, etc., are known as radii vectores, r. The following simple relations hold for all ellipses:

$$\begin{aligned} SA &= a\,(1+e) \\ SP &= a\,(1-e) \\ p &= r\,(1+e\cos\theta) \\ b^2 &= a^2\,(1-e^2) \end{aligned} \qquad \ldots \quad \ldots \quad \ldots \quad (52)$$

Kepler's Second Law

The radius vector joining the sun to a planet sweeps out equal areas in equal times.

This law is illustrated in Fig. 40, where P_1, P_2 and P_3, P_4 are two pairs of points in the orbit of a planet so that the time required for the planet to revolve from P_1 to P_2 is the same as the time required to revolve from P_3 to P_4. The second law states that the area P_1SP_2 is equal to the area P_3SP_4. Since SP_4 is greater than SP_2 it is obvious that the arc P_3P_4 must be less than the arc P_1P_2 to produce the equality in area between P_1SP_2 and P_3SP_4. Hence the greater the distance of a planet from the sun the less the arc it will traverse in a given time, and the nearer it is to the sun the greater the arc it will traverse in the same time. The earth moves over a greater arc in the same time on January 2 when it is nearest to the sun than it does on July 4 when it is at its greatest distance from the sun.

Kepler's Third Law

Kepler's third law is as follows:

The squares of the periodic times of any two planets are in the same proportion as the cubes of their mean distances from the sun.

It has been shown in (52) that $SA = a(1 + e)$, and $SP = a(1 - e)$. SA and SP being the greatest and least distances of a planet from the sun, and hence the mean distance is the arithmetical mean of these two distances, that is, the mean distance is a. If T be the periodic time of the planet, that is, its sidereal year, Kepler's third law asserts that a^3/T^2 is the same for all planets revolving round the sun. It should be noticed that this ratio is independent of the eccentricity of the orbit and depends only on the periodic time and the semi-major axis.

Suppose we apply Kepler's third law to the earth. In this case we can take T to be a sidereal year and a to be an astronomical unit, 93,005,000 miles, and we can use the law to find the mean distance of any other planet in the solar system, provided we know its sidereal period. It is necessary to use the same units throughout, that is, the unit of distance is 1 astronomical unit and the unit of time is 1 sidereal year. Of course we could have used other units. We might have taken a kilometre as the unit distance, or a mile, and 365·224 days as the unit of time, but these would prove very inconvenient. The units suggested are those universally in use, and they will be employed in subsequent calculations. Kepler's third law can be expressed in the form

$$a^3/T^2 = 1 \qquad \ldots \qquad \ldots \qquad \ldots \qquad \ldots \qquad (53)$$

by taking the proper choice of units.

Example 1

As an application of Kepler's third law take the case of the planet Mars whose sidereal period is known to be 686·95 days or 1·881 years. What is its mean distance from the sun?

Since $T = 1{\cdot}881$, expressed in the unit adopted for the time, it follows from (53) that $a^3 = 3{\cdot}538161$, and hence $a = 1{\cdot}524$ astronomical units. If we wish to find this distance in miles it is only necessary to multiply 1·524 by 93,005,000 miles, and the result is 141,740,000 miles. The mean distances of all the planets can be found in a similar manner.

The Most Accurate Determination of the Solar Parallax

If we know the distance of a body comparatively close to the earth and also its sidereal period, we have the data for determining the distance of the sun from the earth and hence the sun's parallax. The planets

Venus and Mars have been used for this purpose, but the best determination of the solar parallax has been made by means of the minor planet Eros, which sometimes comes within 14 million miles from the earth. When it makes a close approach its distance from the earth is found by a process similar to that used in determining the distance of the moon. Knowing the sidereal period of Eros, its mean distance from the sun is also known in terms of the earth's distance from the sun, whatever that may be. We are not concerned with the actual mean distance of the earth from the sun for the moment—we merely take this as one unit and then the mean distance of Eros from the sun is calculated from its sidereal period in terms of this unit.

If the actual distance of Eros from the earth at any time is known in miles or kilometres or any other standard unit and its distance is also known in terms of an astronomical unit (which we wish to express in miles) it is possible to equate the fraction of an astronomical unit denoting the distance of Eros from the earth with its actual distance in miles, and hence to determine the value of an astronomical unit.

The most recent determination of the value of the astronomica unit was accomplished by Sir Harold Spencer Jones, the Astronomer Royal, who completed his investigations in 1941. Eros approached the earth in 1930–31 to within a distance of 16 million miles and twenty-four observatories in different parts of the world co-operated in observing the body. An enormous amount of work was involved in making the necessary reductions and introducing various refinements and corrections to ensure accuracy. Although ten years were spent on this work it amply repaid the labour, as the solar parallax was determined with an accuracy never previously attained, and it does not seem likely that a more accurate determination will be made for many years.

Newton's Law of Gravitation

Kepler's three laws can be deduced from Newton's law of gravitation, which can be stated as follows:

Every particle of matter in the universe attracts every other particle with a force varying directly as the product of their masses and inversely as the square of the distance between them. In the case of a spherical body Newton showed that its attraction on a particle outside the sphere was the same as if the entire mass of the body were concentrated at its centre. If, therefore, we assume that the sun and planets are spherical, which is very nearly true, and that the distance between the centre of the sun of mass M and the centre of a planet of mass m is r, the attraction of the sun on the planet is Mm/r^2, expressed in proper units, and the attraction of the planet on the sun is the same. To express this force of attraction in the usual units, the following definition has been adopted:

The constant of gravitation, denoted by G, is the force in dynes with

which a spherical mass of 1 gram would attract another spherical mass of 1 gram, when the distance between the centres of the two spheres is 1 centimetre. It has been found that the value of G is $6{\cdot}67 \times 10^{-8}$ c.g.s. units, and from our knowledge of the value of this constant it is possible to find the gravitational attraction between any two spherical bodies, provided their masses and also the distance between their centres are known.

The method just described for determining the attraction of one body on another is not rigorously accurate, though in the case of the sun and most of the planets it can be used with accuracy sufficient for all practical purposes. The modification in the form of the expression given above is as follows:

Modification in Kepler's Third Law

Let S and P denote the masses of the sun and a planet respectively, and let r be the distance between their centres. Then, k being a constant, the attraction of the sun on each unit mass of P is kS/r^2, and hence the sun's attraction on the mass P is kSP/r^2. Similarly, the attraction of P on S is kPS/r^2, so that the moving force with which the masses S and P tend towards each other is the same on each body—a necessary consequence of the equality of action and reaction.

The velocities with which the bodies would approach each other are different. The expression for the velocity of P, which would be generated in unit time, is obtained by dividing the force kSP/r^2 by P, and is kS/r^2. Similarly, the velocity of S which would be generated in unit time is kP/r^2, and each of these is a measure of the acceleration due to the action of P and S respectively.

The relative motion of two bodies is unaltered if equal and parallel velocities be given to each one, and hence we can bring the sun to rest, relative to the planet, by giving the sun a velocity kP/r^2 in a direction opposite to that of the force exercised by the planet on the sun. We must apply the same velocity to the planet, and hence when the sun is reduced to relative rest there are two accelerations acting on the planet, kP/r^2 and kS/r^2, so that the total acceleration of the planet towards the sun, regarded as a fixed centre, is

$$k\,(S + P)/r^2.$$

Hence it is necessary to regard the absolute force between the sun and the planet as proportional, not to S, but to $S + P$. This modifies Kepler's third law, but in the case of most of the planets this modification is very small and insignificant. The modification is as follows:

Instead of writing $a^3/T^2 = 1$, the correct form is

$$a^3/T^2 = 1 + P/S \qquad \ldots \quad \ldots \quad \ldots \quad (54)$$

The ratio P/S is $1/1047$ for Jupiter and $1/3502$ for Saturn, while it is only $1/333{,}434$ for the earth, and hence in the latter case a^3/T^2 is altered only very slightly by taking the mass of the earth into consideration.

Computation of the Mass of a Planet

It is possible to find the mass of a planet which has one or more satellites by a slight modification of (54). If s is the mass of a satellite and t and a_1 its sidereal time of revolution round the planet and its semi-major axis respectively, the semi-major axis being the mean distance of the satellite from the planet, (54) can be expressed in the form

$$a_1{}^3/t^2 = C\,(1 + s/P) \qquad \ldots \quad \ldots \quad \ldots \qquad (55)$$

where C is the ratio of the planet-satellite mass to the sun-earth mass. The application of this formula will be shown for the planet Mars.

Example 2

Mars has two satellites revolving round him, the nearest of which—Phobos—has a sidereal period of 0·31801 day or 0·0008731 year. Its mean distance from Mars is 5834 miles, or 0·000062725 astronomical unit. In the case of the Mars-Phobos system, therefore, we can write the constants as follows:

$$t = 8{\cdot}731 \times 10^{-4}, \qquad a_1 = 6{\cdot}2725 \times 10^{-5},$$

the units being the same as those employed in the case of the earth and sun. Hence from (55),

$$(6{\cdot}2725^3 \times 10^{-15})/(8{\cdot}731^2 \times 10^{-8}) = C\,(1 + s/P).$$

From this we find $C\,(1 + s/P) = 3{\cdot}24 \times 10^{-7}$.

The mass of Mars and Phobos is, therefore, $3{\cdot}24 \times 10^{-7}$ that of the earth and sun, or ignoring the mass of Phobos in comparison with that of Mars, and the mass of the earth in comparison with that of the sun, the mass of Mars is $3{\cdot}24 \times 10^{-7}$ that of the sun. Deimos, the other satellite of Mars, can be used in a similar manner to find the mass of Mars, and the same result follows.

The mass of the earth-moon system in comparison with that of the sun-earth system can be found in the same way. The moon's sidereal period

is 0·0748 year and her mean distance from the earth is 0·002571 astronomical unit, and hence (55) gives

$$0{\cdot}002571^3/0{\cdot}0748^2 = C\,(1 + s/P).$$

Hence $C\,(1 + s/P) = 0{\cdot}00000303471$.

This shows that the mass of the earth-moon system is $3{\cdot}03471 \times 10^{-6}$ that of the sun-earth system.

Orbital Velocity of a Planet or a Comet

The velocity V in miles per second of a planet (or comet) at a point in its orbit where its distance from the sun is r can be found from the formula

$$V^2 = \mu\left(\frac{2}{r} - \frac{1}{a}\right) \qquad \ldots \quad \ldots \quad \ldots \quad (56)$$

where μ is a constant for all bodies revolving round the sun. If the planet moves very nearly in a circle, as in the case of Venus, $r = a$ approximately, and (56) becomes

$$V^2 = \mu/r \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (57)$$

The earth moves nearly in a circle and hence (57) holds approximately for the earth. A rigorous value for the planets is given by the expression,

$$V = 18{\cdot}49\,\sqrt{\left(\frac{2}{r} - \frac{1}{a}\right)} \qquad \ldots \quad \ldots \quad (58)$$

In the case of Jupiter, the mass of which is about 0·001 that of the sun, (58) requires slight modification, but this is so small that (58) can be used for all practical purposes for all the planets, including Jupiter, and the comets. In the case of the latter a is usually very large compared with r, and hence $1/a$ is so small that it can often be neglected. In these circumstances

$$V = 18{\cdot}49\,\sqrt{(2/r)} = 26{\cdot}15/\sqrt{r} \qquad \ldots \quad \ldots \quad (59)$$

From (58) it appears that the velocity of a planet in its orbital motion around the sun decreases with increasing distance of the planet from the sun. This is in accordance with Kepler's second law, because the greater the distance of a planet from the sun the greater is the triangle SP_3P_4 in Fig. 40 if the arc P_3P_4 were the same. To balance the greater area due to the greater lengths of the sides SP_3 and SP_4 of the sector SP_3P_4, the

arc P_3P_4 must be smaller the farther the planet is from the sun, and hence the smaller is the velocity of the planet. From this fact the direct and retrograde motions of the planets are easily explained.

Direct and Retrograde Motions of the Planets

Suppose the inner and outer circles in Fig. 41 represent the orbits of the earth and Jupiter, these orbits being supposed to be in one plane which is the plane of the paper. If E and J are the positions of the earth and Jupiter when Jupiter is in opposition, that is, in a line with the sun and the earth, then when the earth is at E' Jupiter will be at J',

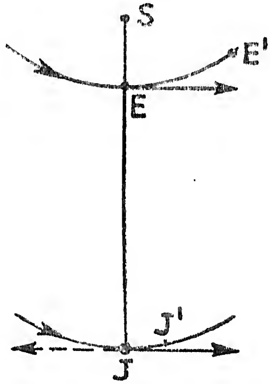

FIG. 41
Explanation of retrograde motion of a planet.

the arc EE' being larger than JJ'. The motion of Jupiter is judged by projecting the planet on the background of stars and when the earth is at E the direction of Jupiter will be EJ. When the earth is at E' the direction of Jupiter will be $E'J'$, and hence an observer on the earth will describe the motion of Jupiter at opposition as retrograde, i.e. in a direction opposite to that of the earth's motion. The lengths of the arcs EE' and JJ' have been exaggerated to show the effect.

In the position shown in Fig. 42 JE is a tangent to the earth's orbit so that the elongation of Jupiter, measured by the angle JES, is 90°. Jupiter is then in quadrature and will no longer appear to have a retrograde motion. While the earth is moving directly away from Jupiter for a very short period, Jupiter will have moved in the same interval to J', and an observer on the earth will see the planet projected on the background of stars in the direction EJ', so that the motion of Jupiter will be direct at quadrature.

The case of a superior planet only has been considered, but the

reader can easily draw diagrams which show the effect in the case of inferior planets also.

It is not surprising that the ancient astronomers, who regarded the earth as fixed, the heavenly bodies all revolving round it, were puzzled by the phenomena of direct and retrograde movements of the planets. They were forced to postulate a movement of each planet round the centre of a circle which in turn revolved round the earth, the names "epicycle" and "deferent" being given to each of these.

If we could imagine an observer on the sun watching the movements of the planets it is obvious that he could tell exactly how long any planet required to revolve round the sun—in other words, he could find the

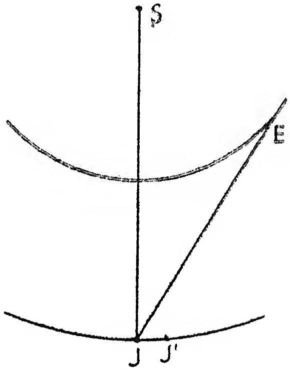

FIG. 42
Explanation of direct motion of a planet.

planet's sidereal period—by noticing how long it took to return to the same position with regard to the stars. It would not be necessary to observe the planet over a complete revolution (an astronomer would require to live nearly 250 years to see Pluto complete its revolution); it would be necessary merely to observe the number of degrees through which the planet moved in a certain time, and as a complete circuit is 360°, to divide 360° by the number of degrees and multiply the result by the time. As an astronomer is unable to observe from the sun he must find a planet's sidereal period by other means.

Synodic and Sidereal Periods of a Planet

When a planet observed from the earth lies in a line between the earth and the sun, it is said to be in *inferior conjunction*. If the sun lies between the earth and the planet, it is said to be in *superior conjunction*. If a planet is in the part of the heavens directly opposite the sun, it is

said to be in *opposition*. The interval between two successive conjunctions or two successive oppositions is known as the planet's *synodic period*, and is the *apparent time* that the planet requires to revolve around the sun. The synodic period is determined by observation, and when it is known it is very easy to find the planet's sidereal period.

Take first of all the case of an inferior planet, that is, a planet whose orbit lies inside that of the earth, like Mercury or Venus. Let P be the sidereal period and S the synodic period, the sidereal period of the earth being E. An observer on the sun would be able to compute the angular velocity of the planet and of the earth as follows.

Assuming uniform motion for each body, the observer on the sun would know that the angle described by the earth in unit time was $360°/E$, and that the angle described by the planet in unit time was $360°/P$. He would not be concerned with synodic periods but an observer on the earth would be, and he could find a simple relation between the planet's synodic period and the sidereal period of each body. The inferior planet traces out a larger angle than does the earth in the same interval and gains $360°/S$ in unit time. Hence we have the relation

$$\begin{aligned} 360/S &= 360/P - 360/E, \text{ or} \\ 1/S &= 1/P - 1/E, \text{ or} \\ 1/P &= 1/S + 1/E \qquad \ldots \quad \ldots \quad \ldots \quad (60) \end{aligned}$$

If the orbit of the planet is outside that of the earth, that is, if we are dealing with a superior planet, the same method is used, but in this case P is greater than E and hence the equation corresponding to (60) is

$$1/S = 1/E - 1/P \qquad \ldots \quad \ldots \quad \ldots \quad (61)$$

The sidereal period of any planet can be found from the expression

$$1/P = 1/E \pm 1/S \qquad \ldots \quad \ldots \quad \ldots \quad (62)$$

the upper sign being used when we are dealing with an inferior planet and the lower sign when we are dealing with a superior planet.

The application of (62) will be illustrated by two examples.

Example 3

The synodic period of Venus is 583·92 days. What is her sidereal period?

Since Venus is an inferior planet, and E = 365·25 days, (62) becomes

$$1/P = 1/365{\cdot}25 + 1/583{\cdot}92 = 1/224{\cdot}70.$$

Hence P = 224·70 days for Venus.

Example 4

In the case of Mars where $S = 779{\cdot}94$ days, (62) gives

$$1/P = 1/365{\cdot}25 - 1/779{\cdot}94 = 1/686{\cdot}95.$$

Hence $P = 686{\cdot}95$ days for Mars.

All the other planets can be dealt with in a similar manner.

Problems

1. The semi-major axis of the orbit of Mars is 1·5237 astronomical units and the eccentricity of his orbit is 0·0933534. Find the length of the semi-minor axis of the orbit and also the greatest and least distances of the planet from the sun.

2. If a minor planet has a sidereal period of 6·7 years what is its semi-major axis?

3. The period of Io, a satellite of Jupiter, is 1·76914 days, and its mean distance from Jupiter is 262,233 miles. From these data find the mass of Jupiter in terms of the mass of the sun.

4. The period of Halley's Comet is approximately 76 years. Find the semi-major axis of its orbit.

5. If the eccentricity of the orbit in 4 is 0·967275, what are the greatest and least distances of the comet from the sun, and what is the speed of the comet in miles per second when its distance from the sun is 1·2 astronomical units?

6. Show that in the second part of 5 no appreciable error would occur if the speed of the comet is supposed to be parabolic. Why could this assumption not be made when the comet is far from the sun—say at aphelion?

7. The mean synodic period of Uranus is 369·66 days. Find the sidereal period of the planet in years.

8. If the sidereal period of Pluto is 247·7 years, find its mean synodic period.

9. In 8 what is the sidereal mean daily motion (in degrees) of Pluto?

10. The sidereal period of Triton—the only satellite of Neptune—is 5·8768 days and its mean distance from Neptune is 219,817 miles. Compare the mass of Neptune with that of the sun.

Chapter IX

THE MOON

A brief outline of the motion of the moon and of certain phenomena associated with this motion is all that can be attempted in this chapter. The motion of the moon is extremely complicated and an adequate treatment of the subject would require a volume to itself. The reader will find in this chapter sufficient for most purposes, but if he wishes to pursue this specialized branch further he can consult more advanced works on the subject.

The Barycentre

The moon moves in an elliptic orbit round the earth, just as the earth moves in an elliptic orbit round the sun, the sidereal period being 27·321661 days. This statement requires slight modification, because the mass of the moon, being 0·0123 that of the earth, cannot be ignored, and hence the centre of gravity of the earth-moon system is not at the centre of the earth but at a distance of $0{\cdot}0123 \times 240{,}000$ or nearly 3000 miles from the earth's centre. This point, known as the *barycentre*, is the focus of the ellipse which the moon describes in its motion and is also the focus of the ellipse which the earth describes in its motion. We have already dealt with binaries which revolve round their common centre of gravity, but the earth and moon can be regarded as akin to a binary system, except that the disparity in their masses is considerably greater than it is in the case of binary stars. The earth and moon are revolving round the barycentre, which is 1000 miles below the earth's surface, and an important effect of this is the apparent displacement of the sun during a month.

Fig. 43 shows that at full moon, when the earth lies between the sun and the moon, the earth's centre is nearer to the sun than is its barycentre by 3000 miles, and at new moon, when the moon is between the earth and the sun, the earth's centre is 3000 miles farther from the sun than the barycentre. It has been shown in Chapter VII that the positions of the sun, measured from observatories on the earth's surface, are referred to the centre of the earth, not to the barycentre, and hence the sun appears to be displaced according to the relative positions of the sun, earth and moon. The angle subtended by a line 3000 miles long

at the sun is $206{,}265'' \times 3000/93{,}005{,}000 = 6''.65$, and hence the sun appears to be about $6''.6$ in front of his average position at the moon's first quarter, and the same amount behind the average place at the last quarter. Careful measurements of the exact displacement have shown that the mass of the moon is about 1/81 that of the earth.

The interval between two successive new moons is called the *lunar*

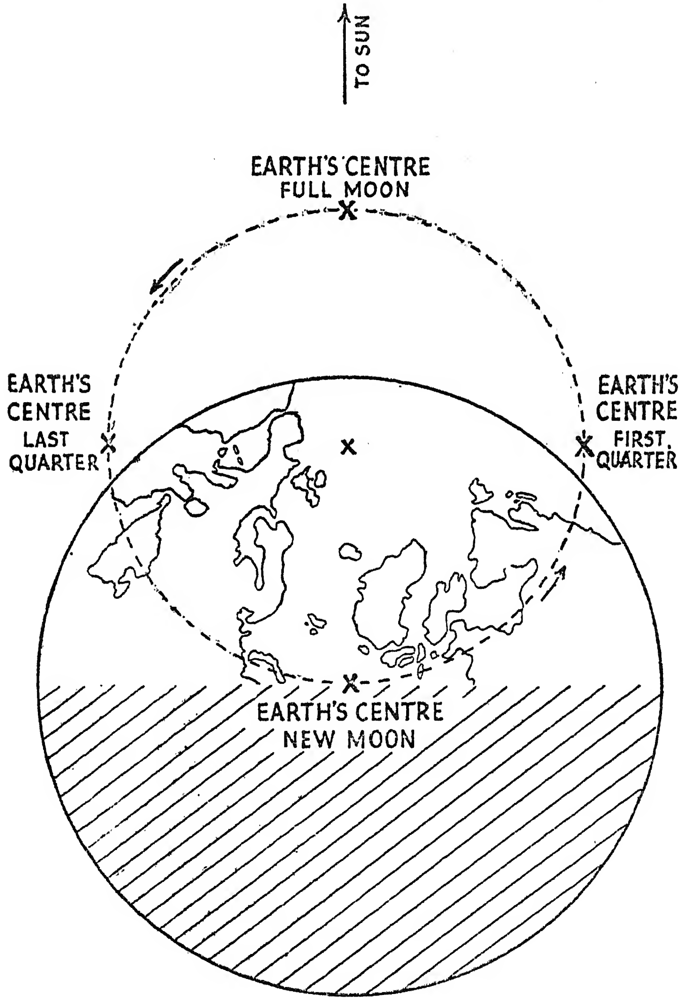

FIG. 43
The earth and moon revolve around their common centre of gravity, known as the barycentre, marked in the figure as the middle **x**.

month or *lunation* or the *synodic period*. Its value varies from month to month, owing to the complexities of the moon's motion, but its average value, taken over a long period, has been found to be 29·53059 days, or $29^d\ 12^h\ 44^m\ 2^s.9$. The sidereal period of the moon, already referred to, is deduced from the observed synodic period in the same way as the sidereal period of a planet is deduced from its synodic period. Taking the sidereal year as 365·25636 days, the synodic month as 29·53059 days, and the moon's sidereal period as M days,

$$1/M = 1/365{\cdot}25636 + 1/29{\cdot}53059, \text{ from which}$$
$$M = 27{\cdot}32166 \text{ days.}$$

New moon occurs at the instant when the centres of the sun and moon are in conjunction, that is, when the centres as seen from the centre of the earth have the same longitude. The age of the moon is the time, expressed in days, that has elapsed since the previous new moon, and when integral values only are used, the moon is said to be one day old when less than 24 hours have elapsed since new moon, two days old during the next 24 hours, and so on. The *N.A.* gives the age of the moon for each day at 0^h G.M.T. throughout the year.

The Metonic Cycle

In 433 B.C. Meton, an Athenian astronomer, made an important discovery regarding the relation between the lengths of the year and a lunation. This relation will be better understood from the table below:

$$19 \text{ tropical years} = 19 \times 365{\cdot}2422 \text{ days} = 6939{\cdot}60 \text{ days almost exactly}$$
$$235 \text{ lunations} = 235 \times 29{\cdot}53059 \text{ days} = 6939{\cdot}69 \text{ days}$$

The difference between the two cycles is only 0·09 day and hence after 19 tropical years the phases of the moon repeat themselves, that is, if it were full moon on a certain date, full moon will occur again on the same date nineteen years later. The Metonic Cycle can be used to predict the dates of full and new moon for many years ahead. A method for applying it will be dealt with later in the book (see pp. 156–58).

It should be noticed that 12 lunations occupy 354·367 days, which is 10·875 days less than the tropical year. If we can imagine new moon occurring at the beginning of a year, then at the beginning of the second year the moon's age will be 10·875 days, at the beginning of the third year 21·750 days, and at the beginning of the fourth year 32·625 days, or, deducting 29·531 days, the period of a lunation, the moon's age at the beginning of the fourth year will be 3·094 days. At the beginning of the fifth year the moon's age will be 13·969 days, and so on.

Inclination of the Moon's Orbit to the Plane of the Ecliptic

The moon does not move in the ecliptic, the plane of her orbit being inclined to the plane of the ecliptic at an angle of 5° 9′ on the average. Hence the moon's orbit intersects the ecliptic in two points known as the *ascending and descending nodes*, the former being applied to the point where she crosses from south to north and the latter to the point where she crosses from north to south. By marking the position of the moon on a globe on a great circle drawn at an inclination of about 5° to the ecliptic the following facts regarding the moon's declination will be obvious.

If the position of the moon coincides with a point on her orbit which is at the maximum distance north from the ecliptic, that is 5° 9′, and this portion of the ecliptic is at its maximum distance north from the equator, about 23½°, then it is possible for the moon to have a declination of more than 28½°. If, on the other hand, the position of the moon on her orbit is at the maximum distance south of the ecliptic, and this happens to be the maximum distance from the equator to the ecliptic, this portion of the ecliptic being south of the equator, the moon's declination will be 28½° S. It is possible, therefore, for the moon to have all declinations between 28½° N. and 28½° S., and this explains why the moon appears so high in the heavens at one time and very low at another time.

Take the case of a full moon about the time of the winter solstice. Since full moon occurs when the earth lies between the sun and the moon, and the declination of the sun at the winter solstice is 23½° S., the moon at this time must have a declination 23½° $\pm$ 5° N. If we take the upper sign this will be 28½° N., and hence in northern latitudes the moon can attain a greater meridian altitude than the sun does at the summer solstice. Take a place in latitude 52° N. At the summer solstice the sun's meridian altitude is colat. + declination (see p. 43) or 61½°, and the moon's meridian altitude at the winter solstice can be 66½°. On the other hand, the altitude might be 5° less than that of the sun at the summer solstice because the declination of the moon when full at the winter solstice might be only 18½° N. In the latter case she attains a meridian altitude of 56½°.

In the summer, when the sun's declination is far north, that of the full moon is far south, and the same reasoning shows that during this season the full moon will be lower when she is on the meridian than the sun is. This brief explanation will show why the altitudes of the moon vary so much throughout the year.

Retardation of the Moon's Transit

If observation be kept on the times when the moon is due south it will be found that she crosses the meridian later each night, but that

there is a considerable variation in the intervals. This variation is due to the fact that the moon does not move at a uniform speed round the earth, as her orbit is eccentric, the eccentricity being 0·0549. The maximum and minimum distances of the moon from the earth are 252,120 miles and 225,880 miles respectively (see p. 95), and at her minimum distance her orbital speed is greater than when she is at her maximum distance. Other factors contribute to irregularities in the motion of the moon, but it is not within the province of this book to deal with these. We are concerned for the present with the retardation, and its *average value* can be easily found as follows.

We have seen that the synodic period is 29·53059 days and in this time the sun crosses the meridian once oftener than the moon. To make this clearer, remember that in this period the moon makes a complete circuit of the heavens, returning to the same position with regard to the sun, and hence if we reckon days by the moon instead of by the sun, there will be only 28·53059 lunar days in 29·53059 solar days. The length of the lunar day is, therefore:

$$29{\cdot}53058/28{\cdot}53059 = 1{\cdot}03505 \text{ solar days.}$$

As 0·03505 solar day $= 50^m\ 28^s$, the interval between transits, *on the average*, is about $50\frac{1}{2}$ minutes.

This retardation explains why the tides are later each day, as the moon is primarily responsible for the tides, the sun acting in a subordinate capacity, owing to his great distance from the earth, which more than offsets his greater mass than that of the moon.

Harvest Moon

If the moon moved along the equator at a uniform rate her times of rising and setting and of crossing the meridian would be later by $50\frac{1}{2}^m$ each day. Not only does the moon not move in the equator; in addition, her motion is far from uniform, and hence considerable variations in the retardation occur, these variations depending on the latitude of the place and other factors. At the full moon nearest to the autumnal equinox it has been observed that the times of rising for a few successive evenings follow sunset at a short interval, and as the continuance of the light is advantageous to farmers for gathering in the harvest, the name *Harvest Moon* has been applied to the moon at this time. It may be remarked that this phenomenon occurs each month but is not so noticeable because it is more conspicuous when this minimum retardation takes place near full moon and also when the moon rises about the time of sunset. At any time when the moon is near ♈ and is moving from the north to the south side of the ecliptic this retardation can be

observed, whatever the phase of the moon, but unless people set out to watch it carefully it will not be very obvious.

To explain this phenomenon, it will assist to refer to Fig. 12 and to suppose that the moon is moving in the ecliptic EE' from S. to N. of the equator. Other circumstances being the same, the change in the moon's declination for any period is greater at the points where the equator and ecliptic intersect than elsewhere. The same thing applies to the sun and a reference to the *N.A.* will confirm this. Thus, on March 23 when the sun is near ♈ the change in his declination each day is more than 23′, whereas the change on May 27 is only 10′ and on June 20 it is about 1′.

Full moon occurs on 1946 September 11^d, and during this date the moon's declination is increasing by about 13′ per hour, whereas a week later the increase is about 3′ per hour at 16^h. The sun is near ♎ on September 11 and hence the moon is near ♈, which accounts for the rapid change in her declination. Readers who possess a celestial globe should measure a few equal intervals on the ecliptic, starting at ♈, and should then measure the declinations of the equidistant points on the ecliptic. It will be found that the declinations increase more quickly near ♈ than they do at a distance from it.

It may seem remarkable that a rapid change in the moon's declination should have any effect on her time of rising and setting. It is to be expected that a change in R.A. would alter these times, an increase of say 50^m in R.A. causing a corresponding delay in her time of rising, transit and setting. The explanation will be easily understood by those who have followed Chapter III. The latitude of Greenwich has been taken in the computations.

By interpolating from the figures given in the *N.A.*, p. 127, it is easily found that at the time of moonrise on September 12, which is near enough to the time of full moon for the present purpose, the moon's R.A. and dec. are $0^h\ 34^m$ and $-\ 1°\ 24'$ respectively, the corresponding figures on September 13 being $1^h\ 25^m$ and $+\ 4°\ 34'$. Making use of (21), Chapter III, the figures for h the hour angle of rising are as follows:

September	*h*	*R.A.*	*Local sidereal time of moonrise*
12	$18^h\ 03^m$	$0^h\ 34^m$	$18^h\ 37^m$
13	17 37	1 25	19 02

The last column is obtained by using (12), Chapter II. It has been shown that the expression

$$\text{local sidereal time} = \text{hour angle} + \text{R.A.}$$

can be used for any heavenly body and it has been applied above in the case of the moon. The difference between the two times of moonrise

is 1 sidereal day 25 minutes, which is the equivalent of about 21^{m} M.S.T. later in the time of rising on the second night under consideration. Hence in this case the moon rises only 21 minutes later on the second night.

Suppose the moon's declination had not changed in the interval or had changed by such a small amount that it was insignificant, what difference would this make in the computation? In these circumstances the moon could be treated as a star, so far as declination is concerned, and in the second row h would be $18^{h}\ 03^{m}$ just as it is on the previous day. Hence the local sidereal time of moonrise on September 13 would be $18^{h}\ 03^{m} + 1^{h}\ 25^{m} = 19^{h}\ 28^{m}$, which differs by 26^{m} sidereal time from the previous figures. This shows the effect of a change in declination on the time of moonrise.

At the full moon following the Harvest Moon the same phenomenon occurs, though it is not generally so pronounced. The moon at this time is called the *Hunter's Moon* because it is the hunting season.

An examination of the *N.A.* will confirm the results just obtained, and certain other interesting matters are shown which are easily explained from the formulae obtained in Chapter III.

It has been shown that the moon is near ♈ on September 12 and actually on September 13 her declination is practically zero, which implies that she is on the equator. We have seen on p. 34 that when a heavenly body is on the equator its times of rising and setting are practically the same for all latitudes, and on referring to the *N.A.*, pp. 518–19, it will be seen that the times of rising and also of setting of the moon on September 13 differ very little for various latitudes. The same applies to other cases where the moon has a small declination, as for instance on May 12.

Now take the case where the moon is near ♈ but not necessarily full, say about May 26. From the explanation given above the moon's declination is changing rapidly at this time and from the *N.A.*, p. 100, it will be seen that the moon is moving north by about 14′ per hour. Hence we should expect that the retardation should be small in the northern hemisphere, and on p. 514 of the *N.A.* it will be seen that this is only 18 minutes in latitude 52° N. For southern latitudes we should expect just the opposite—a considerable retardation—and on p. 530 of the *N.A.* it will be seen that an interval of $1^{h}\ 24^{m}$ exists between moonrise on May 26 and May 27 in latitude 52° S. As the moon is 24 days old and rises about $1^{h}\ 40^{m}$ before the sun in latitude 52° N. the phenomenon is not conspicuous.

It should be noticed that in all cases where the retardation of moonrise is small the retardation of moonset is large, and vice versa. The explanation of this is given below.

Take the case of moonset on September 13 and 14. Interpolating from the *N.A.* figures on p. 127 it is found that at moonset on the above dates the R.A. and dec. of the moon are as follows:

September		R.A.		Dec.	
13		1^h	00^m	$+1°$	39
14		1	55	+7	52

Applying (21) to determine h, the following figures are obtained:

September	*h*		*R.A.*		*Local sidereal time of moonset*	
13	6^h	08^m	1^h	00^m	7^h	08^m
14	6	40	1	55	8	35

The interval between the two times of moonset in this case is $1^d\ 1^h\ 27^m$ sidereal time, which is the equivalent of $1^h\ 23^m$ M.S.T. on the second day.

The reason for the large retardation in the time of moonset will be seen from the two sets of figures—those for moonrise and those for moonset. In the former case the hour angle on the second day is less than on the first day, and this implies that, as the R.A. on the second day is necessarily greater than it is on the first day, a partial compensation is effected by the addition of the less hour angle to the increased R.A. In the case of moonset both the hour angle and the R.A. are greater on the second day than they are on the first day, and hence their addition, giving the local sidereal time of moonset, does not effect a partial compensation but accentuates the retardation.

The effects of refraction and of parallax have been ignored and the times of rising and setting are considered to occur when the centre of the moon's disc is on the horizon. The moon is actually considered to rise and set when her upper limb is on the horizon, like the sun, but the neglect of these points makes no difference to the argument and does not seriously affect the quantitative results.

The phenomena just described are very simply explained by the use of a celestial globe which can be set for any convenient northern latitude —say about 50° N. Imagine that the moon is at ♈ where her R.A. and dec. are zero. Rotate the globe eastward until the moon is on the horizon. The hour angle of rising is measured by the arc from the meridian, round the equator in a westerly direction to the moon, and is obviously equal to 24^h—the arc from the meridian to the moon measured in an easterly direction. The latter is, of course, easier to measure and it will be found that it is 6^h, and hence the hour angle of the moon at rising is 18^h. If the globe is rotated until the moon is on the horizon again at the time of setting, the hour angle in this case is 6^h, assuming that the moon has not moved in R.A. or dec.

Now instead of taking the moon on the equator, imagine that she is a few degrees north of the equator, her R.A. still being zero. When the globe is rotated so that the moon is on the horizon at rising it is found that the angle from the meridian to ♈ is greater than 6^h and hence h is

less than 18^h when the moon is rising. Rotating the globe until the moon is on the horizon again at the time of setting, the angle from the meridian to the moon exceeds 6^h and in this case h is greater than 6. These results were brought out in the above investigation.

Instead of making the moon move northwards in declination make her move south and notice that precisely the opposite phenomena now occur. At the time of rising her hour angle has increased and at the time of setting it has decreased. Hence to observers in the southern hemisphere a large retardation in the time of the moon's rising would correspond to a small retardation to observers in the northern hemisphere.

It should be noticed that in these experiments with a globe the R.A. has been maintained constant as the object is to show the effect of changes in the moon's declination on the times of her rising and setting.

The Moon's Librations

The moon rotates on her axis in the same time as her sidereal orbital period of 27·3317 days and hence presents practically the same face towards the earth. If her axis of rotation were perpendicular to the plane of her orbit and the orbit were circular so that the orbital motion was uniform, we should be able to see just a very little more than half her surface. This is due to the fact that as the moon is a comparatively close body, observers on different parts of the earth see a little more than half of her surface. An observer at any particular place sees this because he is carried round by the rotation of the earth, but this effect is small in comparison with two other effects which will now be considered.

Just as the earth is sometimes ahead of and behind the positions it would occupy if its angular orbital motion round the sun were uniform, so the moon, owing to her elliptical motion round the earth, is sometimes ahead of and behind her mean position. Assuming a uniform axial rotation of the moon, it is obvious that additional portions of her surface are seen on her east and west limbs. This phenomenon is known as *libration in longitude.*

The axis of rotation of the moon is not perpendicular to her orbital plane but is inclined at an angle of $83\frac{1}{2}°$ to this, or at an angle of $6\frac{1}{2}°$ to the perpendicular to this plane. The result is that portions of the moon's surface on "the other side of the moon" are visible, these portions extending $6\frac{1}{2}°$ beyond the moon's poles. This effect is known as *libration in latitude.*

In consequence of these three librations about 59% of the moon's total surface is visible from the earth.

Eclipses of the Sun and Moon

An eclipse of the sun occurs when the earth enters the shadow cast by the moon and so can take place only when the moon is between the

earth and the sun—that is at new moon. Fig. 44 shows the shadow cast by the moon, and this is a cone whose vertex is O. The portion shown in dark shading is the *umbra*, inside of which no light from the sun can pass. Outside the umbra is the *penumbra*, shown in light shading, and some of the light of the sun enters this portion. Transverse tangents from the sun to the moon enclose this space on the side of the moon remote from the sun.

On any part of the earth between P and P' the eclipse will be total, but outside these points the surface of the earth will be in the penumbra, and a partial eclipse will be visible under these conditions. As will appear in the course of the investigation, a total eclipse of the sun is visible over a very small part of the earth.

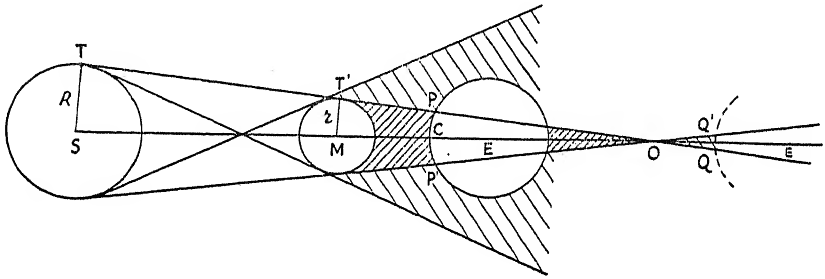

FIG. 44
An eclipse of the sun.

In Fig. 44 let R and r be the radii of the sun and moon respectively, and let O be the vertex of the cone formed by the tangents to the sun and moon. From the properties of similar triangles,

$$SO/OM = R/r = 400.$$

Now $SO = SM + MO$, hence

$$SO/OM = SM/OM + 1, \text{ from which}$$
$$SM/OM = 399, \text{ or}$$
$$OM = 0{\cdot}0025063\ SM.$$

If SM is 93,000,000 miles, OM is about 233,000 miles.

The distance OM of O from the moon's centre will vary according to the value of SM, but the above can be taken as its mean value.

At any portion of the earth's surface between P and P' there will be a total eclipse of the sun. The mean distance between the centres of the earth and moon is about 240,000 miles, but P can be nearly 4000 miles nearer to M than this, though this would not occur frequently as it requires that the moon should be in the zenith of P. Assuming that the

moon is in the zenith of P, this implies that P is 236,000 miles from M and hence would lie outside the vertex O of the cone. This shows that if the moon had been a little further off from the earth total solar eclipses could not have occurred. When the moon is at her greatest distance from the earth—252,120 miles—the point P might possibly be just over 248,000 miles from M the centre of the moon, and in such circumstances a total solar eclipse would be impossible. When the moon is at her least distance from the earth—225,880 miles—the point P could be 221,900 miles from M, and as this is well under the limiting mean value of the distance of the vertex of the cone from the moon's centre a total eclipse is possible.

Annular Eclipses

If the points P and P' on the surface of the earth fall beyond the vertex O, say at Q and Q', inside the portion QQ' the eclipse will appear annular.

Width of the Shadow during a Total Eclipse

From similar triangles, MOT', COP, considering the small arc CP to be a straight line of length s,

$$CP/CO = MT'/MO, \text{ or } s = CO \cdot MT'/MO.$$

Suppose that $MO = 233{,}000$ miles and $MC = 221{,}800$ miles, then CO is 11,200 miles, and hence $s = 1080 \times 11{,}200/233{,}000 = 52$ miles. The width of the shadow is $2s$ and hence is about 100 miles. This occurs under very favourable conditions when the moon is in perigee, but the width of the shadow during totality is often much less than this. The shadow on a small portion of the earth's surface would be a circle of radius 52 miles under the above conditions, if it were projected perpendicular to the horizon at the place. As this does not occur very often, the outline of the shadow is an ellipse, the axis minor being 104 miles with the circumstances as given above, but this is only a particular case and the axis minor of the ellipse is often less than that just indicated.

The calculations required for the circumstances of an eclipse, time, line of totality, etc., are too abstruse to be dealt with in this book. The *N.A.* for each year contains all the details and should be consulted by those who are interested in eclipses. It may be added that there will not be a total eclipse of the sun visible in the British Isles until 1999.

Lunar Eclipses

An eclipse of the moon occurs when the earth is between the sun and the moon, the shadow in this case being cast by the earth, and it can be either total or partial. There is no such thing as an annular

eclipse of the moon. Fig. 45 shows the moon in the umbra and later in the penumbra, these terms being the same as in the case of a solar eclipse. We have seen that the moon's orbital plane is inclined to the ecliptic at an angle of over 5°, and because of this inclination an eclipse of the moon and also of the sun (though not necessarily total) does not take place every month. If the moon is at or close to one of its nodes (see p. 132) and is new or full at the time, an eclipse of the sun or moon will occur.

It has been shown that the distance of the vertex of the umbral cone during a solar eclipse is about 233,000 miles from the centre of the moon and also that this depends on r, the moon's radius. If we substitute the earth's radius for that of the moon we shall obtain the distance of the vertex of the umbral cone when the earth casts a shadow, and as

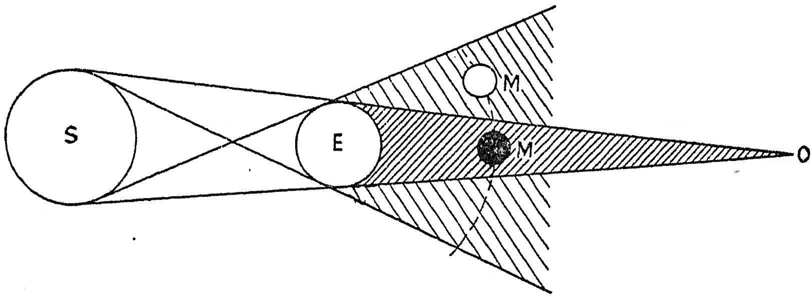

FIG. 45
An eclipse of the moon.

the earth's radius is nearly 3·7 that of the moon, this distance is about 862,000 miles. Hence the vertex O of the cone lies a long way outside the greatest distance of the earth from the moon, and for this reason the moon can never enter the portion of the shadow on the other side of the vertex. Hence an annular eclipse of the moon is impossible.

The moon's nodes are not fixed points on the ecliptic but have a backward movement, making a complete circuit of the ecliptic in 6793·5 days or about $18\frac{2}{3}$ years. It is easily found from this by a method similar to that used in deriving (62), Chapter VIII, that the synodic period of the moon's nodes, that is, the interval between successive coincidences with the sun, is 346·62 days, and from this an important discovery has been made regarding the recurrence of eclipses.

The Chaldean Saros

From the above figures and also from the period of a lunation the following figures are obtained:

19 synodic periods = 6585·78 days
223 lunations = 6585·32 days

The interval of 6585 days is known as the *Saros* and is very important in connection with eclipses. Suppose an eclipse of the sun occurs on May 29, 1919. The moon must have been new at the time and close to one of her nodes, the sun also being close to the same node. In 19 synodic periods the sun must be close to the same node again and in 223 lunations the moon must also be close to the same node, and, in addition, must be new. Hence, the conditions are very nearly the same as for an eclipse 6585 days later and in fact another total eclipse of the sun took place on 1937 June 8, which is 6585 days after May 29, 1919.

The Chaldean astronomers discovered the *Saros* and were able to predict eclipses of the sun and moon by making use of it.

Without giving a proof the following facts about eclipses can be accepted:

During any year there must be at least two eclipses, both of the sun.

During any year there cannot be more than seven eclipses. Of these, four can be solar and three lunar, or five can be solar and two lunar.

From the last rule it will be seen that under no circumstances can there be four lunar eclipses in a year. The word "eclipses" includes every form of eclipse, total, partial, or, in the case of the sun, annular.

Chapter X

THE STARS

Certain problems arise in dealing with the stars, such as magnitudes, proper motions, the masses of binaries, etc., and a brief outline of the method of attacking some of the problems by elementary mathematics follows.

Hipparchus, born at Bithynia about 140 B.C., compiled the earliest star catalogue and divided the stars into six classes according to their brightness. He included about 20 of the brightest stars in the first magnitude and the large number of faint stars that were just visible to the naked eye in the sixth magnitude. Between these extremes, stars of intermediate brightness were catalogued as magnitudes 2, 3, 4 and 5. The higher the number denoting the magnitude of a star the fainter is the star.

Stellar Magnitudes

In 1850 an English astronomer named Pogson established a definite light ratio between the stars of different magnitudes. Suppose we assume that the light ratio between a star of magnitude 1 and a star of magnitude 6 is 100, and that each step in the five steps between magnitudes 1 and 6 is x, we can determine the value of x as follows:

A star of magnitude 1 is x times as bright as a star of magnitude 2, and a star of magnitude 2 is x times as bright as a star of magnitude 3, and so on. Hence a star of magnitude 1 is x multiplied by x or x^2 times as bright as a star of magnitude 3, and so for other magnitudes. In other words, if we want to find how many times a star of magnitude 1 is brighter than a star of magnitude 6, we deduct 1 from 6 and raise x to the corresponding power. Hence a star of magnitude 1 is x^5 times brighter than a star of magnitude 6. We have seen that the interval between magnitudes 1 and 6 has been arbitrarily divided so that the light ratio is 100, and hence we obtain the equation

$$x^5 = 100.$$

Taking logarithms of both sides,

$$5 \log x = 2, \text{ from which } \log x = 0{\cdot}4, \text{ or } x = 2{\cdot}512.$$

Hence, to compare the brightness of two stars we find the difference in their magnitudes and raise 2·512 to the corresponding power, remembering that the brighter star has always the smaller magnitude. Refinements in determining stellar magnitudes have necessitated the use of intermediate numbers. Thus, the magnitude of Regulus is given as 1·34, of Spica 1·21, and so on for other stars.

Various methods are used for finding the ratio between the brightness of stars, but it is not within the scope of a mathematical treatise to describe these. If we take l_1 and l_2 to be the brightness of two stars of magnitude m_1 and m_2 respectively, it is obvious from what has just been said about the equation connecting magnitudes with brightness that

$$k\,l_1 = 1/2{\cdot}512^{m_1}, \quad k\,l_2 = 1/2{\cdot}512^{m_2},$$

k being a constant, and hence

$$l_1/l_2 = 2{\cdot}512^{(m_2-m_1)} \quad \ldots \quad \ldots \quad \ldots \quad (63)$$

The constant k disappears from this and similar computations and hence will not be used in the future as it can be assumed equal to 1.

Taking logarithms of both sides

$$\log \frac{l_1}{l_2} = 0{\cdot}4\ (m_2 - m_1), \text{ or}$$

$$\log \frac{l_2}{l_1} = 0{\cdot}4\ (m_1 - m_2) \quad \ldots \quad \ldots \quad (64)$$

As an example of the application of (64), take the case of the two stars just mentioned, Regulus and Spica. Using the figures for the magnitudes, what is the ratio of the brightness of Spica to that of Regulus?

Let m_1, l_1 denote the magnitude and brightness of Regulus and m_2, l_2 those of Spica. Then, since $m_1 - m_2 = 0{\cdot}13$, $\log \frac{l_2}{l_1} = 0{\cdot}40 \times 0{\cdot}13 = 0{\cdot}052$. Hence $l_2/l_1 = 1{\cdot}27$, or Spica is 1·27 times as bright as Regulus.

If we know the relative brightness of two stars we can find the difference in their magnitudes from (64). Thus, if we were informed that Sirius was 6·67 times brighter than Procyon, and we were asked to determine the difference in their magnitudes, we proceed as follows, l_1 and m_1 applying to Sirius and l_2 and m_2 to Procyon.

$$l_1/l_2 = 6{\cdot}67, \text{ and } \log 6{\cdot}67 = 0{\cdot}824, \text{ hence we have}$$
$$0{\cdot}824 = 0{\cdot}4\ (m_2 - m_1), \text{ from which}$$
$$m_2 - m_1 = 2{\cdot}06, \text{ or } m_2 = m_1 + 2{\cdot}06.$$

Hence the magnitude of Procyon is 2·06 greater than that of Sirius. The magnitude of Sirius is $-$ 1·58, so that of Procyon is 0·48.

A star of magnitude 1 is not the brightest of stars. The star Aldebaran has been assigned a magnitude 1·06 on the scale of visual magnitudes in use, but there are stars much brighter than Aldebaran, and for some of these negative magnitudes, and for others not quite so bright, fractional magnitudes, are necessary. This explains why Sirius—the brightest star in the heavens—has a magnitude − 1·58. The accuracy of the figures can be checked as follows.

It has been found that the brightness of Sirius is 11·37 times that of Aldebaran, and hence, if m_1 is the magnitude of Sirius and m_2 of Aldebaran, then, since log 11·37 = 1·056, (64) gives

$$0{\cdot}4\,(m_2 - m_1) = 1{\cdot}056, \text{ from which}$$
$$m_2 - m_1 = 2{\cdot}64, \text{ or } m_1 = m_2 - 2{\cdot}64.$$

Hence the magnitude of Sirius is 2·64 less than that of Aldebaran, and its magnitude is, therefore, 1·06 − 2·64 = − 1·58.

The following problem is a little more difficult than those just considered, and the reader should follow the method used, as questions of this nature will be set in the Examples at the end of the chapter.

The star Castor, which appears single to the naked eye, is resolved by the telescope into two stars of magnitudes 1·99 and 2·85. What is the magnitude of the combined system?

Let l_1 and l_2 be the brightness of each component, l the brightness of the combined system and m its magnitude. Then

$$l_1 = 2{\cdot}512^{-1.99} \qquad l_2 = 2{\cdot}512^{-2.85} \qquad l = l_1 + l_2$$
$$\log l_1 = -1{\cdot}99 \times 0{\cdot}4 = -0{\cdot}796 = -1 + 0{\cdot}204$$
$$\log l_2 = -2{\cdot}85 \times 0{\cdot}4 = -1{\cdot}14 = -2 + 0{\cdot}86$$
$$l_1 = 0{\cdot}160 \qquad l_2 = 0{\cdot}0724$$
$$l_1 + l_2 = 0{\cdot}2324$$
$$2{\cdot}512^{-m} = 0{\cdot}2324, \text{ and hence}$$
$$-0{\cdot}4\,m = \log 0{\cdot}2324 = -1 + 0{\cdot}3662 = -0{\cdot}6338,$$
$$\text{from which } m = 1{\cdot}58.$$

The brightness of a star does not necessarily supply us with any information on its *intrinsic brightness*. If two stars have the same intrinsic brightness but one is further off than the other, the former will appear fainter, or it will have a greater magnitude than the latter. To compare the intrinsic brightnesses of stars, or their *luminosities*, we must compare their brightness when they are at the same distance from us. The standard distance selected for this purpose is 32·6 light-years, which is ten times the distance corresponding to a parallax of one second, and hence is 10 parsecs (see p. 103).

The intensity of illumination varies inversely as the square of the distance that the star is away from us. Hence if the luminosity of a star

is l_1 when its distance is L light-years and its luminosity is l_2 when it is at a distance 32·6 light-years, we have the relation

$$(L/32{\cdot}6)^2 = l_2/l_1$$

The ratio l_2/l_1 is $2{\cdot}512^{(m-m_a)}$ where m is the apparent magnitude of the star and m_a is its magnitude at a distance of 32·6 light-years, or its *absolute magnitude.*

Hence

$$(L/32{\cdot}6)^2 = 2{\cdot}512^{(m-m_a)}$$

Taking logarithms of both sides and remembering that $\log 2{\cdot}512 = 0{\cdot}4$, we obtain the relation

$$2\ (\log L - \log 32{\cdot}6) = 0{\cdot}4\ (m - m_a)$$

Substituting 1·5132 for log 32·6 and simplifying, we obtain

$$m_a = m + 7{\cdot}566 - 5 \log L \qquad \ldots \qquad \ldots \qquad (65)$$

The parallax p of the star can be used instead of its distance in light-years. It has been shown on p. 103 that $L = 3{\cdot}26/p$, and if this value for L be used we have

$$\left(\frac{1}{10p}\right)^2 = 2{\cdot}512^{(m_a-m)}$$

Hence,

$$0{\cdot}4\ (m \quad - m) = -2 \log 10 - 2 \log p, \text{ from which}$$

$$m_a = m + 5 + 5 \log p \qquad \ldots \qquad \ldots \qquad (66)$$

Both of the above formulae will be used to find the absolute visual magnitude of β Centauri, the apparent visual magnitude of which is 0·86, parallax 0″·011, and distance in light-years 296 years.

From (65)

$$m_a = 0{\cdot}86 + 7{\cdot}566 - 5 \log 276 = 0{\cdot}86 + 7{\cdot}566 - 12{\cdot}356 = -3{\cdot}9$$

From (66)

$$m_a = 0{\cdot}86 + 5 + 5 \log 0{\cdot}011 = 5{\cdot}86 - 10 + 0{\cdot}2070 = -3{\cdot}9$$

Relation Between the Effective Temperature, Diameter and Absolute Magnitude of a Star

There is a useful formula connecting the effective temperature of the surface of a star with its diameter and absolute magnitude, the

diameter being computed in terms of the diameter of the sun as the unit. The formulae is fairly good up to temperatures of 7,000° K., but after that it gives only approximate results. This formula is as follows:

Let D be the diameter of the star, that of the sun being the unit, T its absolute temperature, and m_a its absolute magnitude. Then

$$\log D = 5900/T - 0{\cdot}2\, m_a - 0{\cdot}02 \qquad \ldots \qquad \ldots \qquad (67)$$

As an example, take the case of Aldebaran, the temperature of which is 3300° K. and parallax 0."057. To find its diameter we must first of all compute its absolute magnitude by (66), taking its apparent visual magnitude as 1·06.

$$m_a = 1{\cdot}06 + 5 + 5 \log 0{\cdot}057 = 6{\cdot}06 + 5\,(-2 + 0{\cdot}756) = -\,0{\cdot}16$$

From (67)

$$\log D = 5900/3300 - 0{\cdot}2 \times -0{\cdot}16 - 0{\cdot}02 = 1{\cdot}788 + 0{\cdot}012 = 1{\cdot}800$$

$$\text{Hence } D = 63.$$

The sun's diameter is 864,000 miles, and hence the diameter of Aldebaran is about 54 million miles.

Cepheid Variables

There is an important relation between the apparent magnitudes of the Cepheid variables and their period of variation. This relation was discovered in 1912 by Miss Henrietta S. Leavitt, Harvard Observatory, and it enables us to find the distance of a Cepheid variable when its period is known (and this is merely a matter of observation) and also its apparent magnitude.

First of all, the absolute magnitude of the Cepheid must be determined from its period, and this can be done by using the period-luminosity curve, Fig. 46, in which absolute magnitudes are plotted against the logs of the period. Thus, if the period is 100 days so that $\log P = 2$, the absolute magnitude is $-\,6{\cdot}5$. If the period is 0·56 day then, since $\log 0{\cdot}56 = \bar{1}{\cdot}75 = -\,0{\cdot}25$, the curve shows that the absolute magnitude is $-\,0{\cdot}15$. To find the distance of the Cepheid in the latter case, assuming that the apparent magnitude is 15, use (66),

$$-\,0{\cdot}15 = 15 + 5 + 5 \log p,$$

Hence $5 \log p = -\,20{\cdot}15$, or $\log p = -\,4{\cdot}03 = \bar{5}{\cdot}97$, from which $p = 0''000093$, which corresponds to $3{\cdot}26/0{\cdot}000093 = 35{,}000$ light-years.

The curve is difficult to read for small values of the period, and in such cases, where the period is less than a day, the following empirical formula will suffice for all practical purposes:

$$m_a = -0{\cdot}39 - 0{\cdot}95 \log \text{period}.$$

Readers can check the above result by means of this formula, but it should not be used for values of the period much greater than a day.

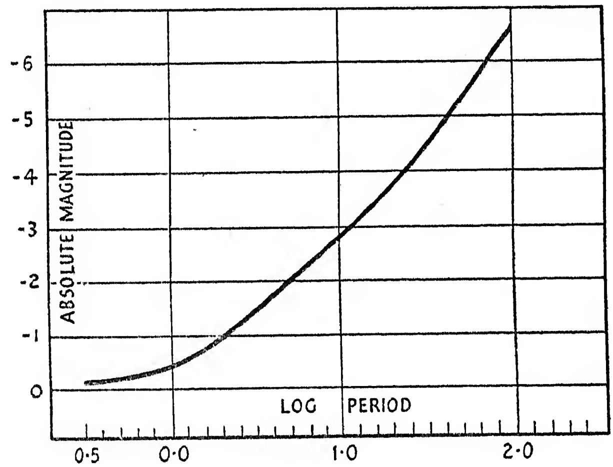

FIG. 46

The relation between the absolute magnitude of a Cepheid variable and the logarithm of its period in days can be taken from the curve. On the left of the zero 0.0 on the horizontal line (the abscissa) the logarithms are negative, the periods in these cases being a fraction of a day.

Masses of Binary Systems

It has been shown on p. 123 that the mass of a planet can be found when the distance of a satellite from the planet and also its period of revolution are known. The same method can be used to determine the mass of a binary system—not the mass of each component of the system but the combined mass of the two stars. Writing (54) in the form

$$a^3/T^2 = S + P,$$

taking the mass of the sun as the unit, and applying the same formula to a binary system in which the mass of each component is m_1 and m_2, the semi-major axis of their orbit a astronomical units, and their period

of revolution round their common centre of gravity T years, then we obtain the expression

$$a_1^3/T_1^2 = m_1 + m_2 \quad .. \quad .. \quad .. \quad (68)$$

Before applying (68) it is necessary to know the distance of the system from the earth or the sun (it is immaterial which is used because this distance is so great that it can be considered the same whether it is measured to the earth or the sun). There is no necessity to find the distance in light-years or astronomical units; it is sufficient to determine the parallax of the system. Neither is it necessary to determine the length of the semi-major axis of the orbit described by the system; this can be measured in seconds of arc and the result used as shown in the following formula.

If d is the distance of the system from the sun, measured in astronomical units, then, p being the parallax of the binaries,

$$\sin p = 1/d \quad .. \quad .. \quad .. \quad .. \quad (69)$$

If α is the angle in seconds of arc subtended at the earth (or the sun) by the semi-major axis of the system, then

$$\sin \alpha = a_1/d \quad .. \quad .. \quad .. \quad (70)$$

Dividing (70) by (69) and remembering that both p and α are very small angles, so that the angles can be substituted for their sines,

$$a_1 = \frac{\alpha}{p}$$

Hence we can substitute α/p for a_1 in (68) and the result is

$$m_1 + m_2 = \left(\frac{\alpha}{p}\right)^3 \Big/ T^2 \quad .. \quad .. \quad .. \quad (71)$$

The application of (71) will be shown for Sirius and its companion, the data being as follows:

The parallax p of Sirius is $0''.371$, the semi-major axis α of the system is $7''57$, and the period T of revolution is 50 years. What is the mass of the binary system in terms of the mass of the sun as the unit?

By (71)

$$m_1 + m_2 = \left(\frac{7{\cdot}57}{0{\cdot}371}\right)^3 /2500 = 3{\cdot}40.$$

Proper Motion of a Star

The proper motion of a star, which is the rate of change in the position of the star on the celestial sphere, relates to that part of the motion which is transverse to the line of sight. The term radial velocity is applied to that part of the motion which is towards or from us, but this cannot be detected by observing the position of the star with reference to the background of stars after the lapse of a certain time. The spectroscope is used to measure the radial velocity, the method of application being as follows.

If $\triangle\lambda$ is the change of wave-length of a line in the spectrum of the star, the wave-length of the line being λ, the radial velocity is

$$\textit{Velocity of light} \times \triangle\lambda/\lambda$$

The star is approaching or receding according as the displacement is towards the violet or red end of the spectrum. Thus, if the wave-length of a line in the spectrum of a star is 4861·102 and the wave-length of the line in the comparison spectrum is 4861·327, the radial velocity of the star away from the earth is 0·225/4861 or 0·0000463 the velocity of light. Hence the radial velocity is 13·9 kilometres, or about 8·7 miles a second.

If μ is the annual proper motion of a star whose parallax is p, and we require its tangential velocity in miles a second, this cannot be accomplished until we know the distance of the star. If the parallax of the star is p, its distance is $3{\cdot}26/p$ light-years, or $19{\cdot}2 \times 10^{12}/p$ miles. The number of miles traversed by the star at right angles to the line of sight is therefore $19{\cdot}2\,\mu \times 10^{12}/206{,}265\,p$ in a year, and dividing this by the number of seconds in a year, the result is $2{\cdot}94\,\mu/p$ miles or $4{\cdot}74\,\mu/p$ kilometres. Hence if $\triangle$ is a star's tangential velocity,

$$\triangle = 2{\cdot}94\,\mu/p \text{ miles} = 4{\cdot}74\,\mu/p \text{ kilometres per second} \quad .. \quad (72)$$

Take the case of Kapteyn's star, which has an annual proper motion of 8″.76. Its parallax is 0″.317, and hence its tangential velocity by (72) is

$$2{\cdot}94 \times 8{\cdot}76/0{\cdot}317 = 81 \text{ miles a second.}$$

When the radial velocity V and the tangential velocity T are known, the space velocity v of the star is easily found from the formula

$$v^2 = V^2 + T^2 \quad .. \quad .. \quad .. \quad (73)$$

the well-known formula for the parallelogram of velocities. If the direction of motion makes an angle θ with the line of sight, θ is derived from the formula,

$$\tan \theta = T/V \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (74)$$

Arcturus has a radial velocity of 3 miles a second away from us and a tangential velocity of 84 miles a second. Its space velocity, derived from (73) is therefore 84·05 miles a second. The direction which its space motion makes with the line of sight by (74) is $\tan^{-1} 84/3 = \tan^{-1} 28 = 88°$.

Problems

1. The apparent visual magnitude of α Centauri is 0·06 and of α Leonis 1·34. Compare the brightness of α Centauri with that of α Leonis.

2. The apparent visual magnitude of α Carinae is $-$ 0·86 and of α Virginis 1·21. Compare the brightness of α Carinae with that of α Virginis.

3. The apparent visual magnitude of α Aurigae is 0·21 and of α Eridani 0·60. Compare the brightness of α Eridani with that of α Aurigae.

4. The apparent visual magnitude of α Bootis is 0·24 and its parallax is 0″·080. Find its absolute visual magnitude.

5. What is the absolute visual magnitude of α Aquilae, apparent visual magnitude 0·89, if its parallax is 0″·204?

6. If two stars differ in magnitude by 2·34, compare their brightness viewed in the telescope.

7. The star α Crucis is a double star, the magnitudes of the components being 1·58 and 2·09. What is the apparent visual magnitude of the system as seen by the naked eye?

8. ζ Ursae Majoris seen with the telescope is a double star, the magnitudes of the pair being 2·40 and 4·50. What is the magnitude of the star as seen by the naked eye?

9. The magnitude of the sun is $-$ 26·72. How many times does the brightness of the sun exceed that of a first magnitude star?

10. Taking the sun's apparent visual magnitude as given in 8, find his absolute magnitude.

11. What is the diameter of α Aurigae if its absolute temperature is 5500°, apparent visual magnitude 0·21, and parallax 0″·069?

12. The period of a Cepheid variable is 12·6 days and its apparent magnitude is 14·5. Find its distance in light-years.

13. The star α Geminorum consists of two components which revolve

round the common centre of gravity of the system in 306·3 years. The semi-major axis of the orbit is 6″.06 and the parallax of the system is 0″.076. Find the combined mass of the system, taking the mass of the sun as the unit. What is the mass in grams?

14. The star α Centauri, parallax 0″.758, is a binary, the semi-major axis of the system being 17″.67, and the period of revolution 80 years. Compare the mass of the system with that of the sun.

APPENDIX I

On Finding the Path of a Meteor from Two or More Observations

This method resembles that used by the surveyor for measuring distances by using a base line and taking two angles. It is a little complicated owing to the fact that we are working in three dimensions, and other complications arise from faulty observations, about which something will be said later.

Suppose an observer at *O* sees a meteor dashing across the heavens and records its position at the beginning and end of its flight, such positions being noticed with reference to some stars. Knowing the right ascension and declination of the stars, he can then say that the meteor moved from a certain right ascension and declination to another right ascension and declination on a certain date at a certain time, and, unless the meteor has a very high speed, he can estimate approximately its time of flight, say 2 or 3 seconds. This latter is not really essential for finding the path of the meteor, but is useful if its speed is to be found. Another observer at *O′*, 20 or 30 miles or even more away from *O*, also observes the same meteor, and he records its positions too, but, of course, these will be entirely different from those recorded by *O*.

The azimuths and altitudes of the meteor for the beginning and end of its path must be computed for the positions of each observer. The necessary formulae for deriving these are given in Chapter III, but a globe can be used instead of formulae because extreme accuracy is generally unnecessary in finding azimuths and altitudes for meteor work. An error of one or two degrees will not affect the results very seriously, and a globe is much speedier than the use of formulae. The method for finding altitudes and azimuths by means of a globe is given on pp. 50–2.

A specific example will now be taken to show the method adopted.

Suppose that an observer at Hastings sees a meteor and that the subsequent computation or the use of a globe gives its azimuth and altitude at the beginning of its flight as 50°W. and 55° respectively. An observer at Maidstone sees the meteor at the same instant and from his observation its azimuth is found to be 87°W. and its altitude 64°.

Take a map of England and lay one end of a rule on Hastings. With the aid of a protractor lay the rule along an azimuth of 50°W. Do the

same with Maidstone, so that the rule there runs at an azimuth of 87°W. It will be better to draw faint lines on the map which can be subsequently erased without spoiling the map, but the rules will suffice in most cases. Notice that the lines intersect near Epsom, and this shows that the commencement of the flight of the meteor was directly over Epsom. The next step is to find the height of the meteor above the earth at Epsom.

Measuring the distance from Epsom to Hastings we find that it is 49 miles, and also the distance from Epsom to Maidstone is 34 miles. It is obvious that the height of the meteor must be 49 tan 55° or 34 tan 64°, each of which is just under 70 miles, which is, therefore, the height of the meteor when it was first observed.

In a similar manner its position and height are found from the observations at the end of its flight. Suppose that this shows it ended over Chelmsford at a height of 30 miles. The distance from Epsom to Chelmsford is 42 miles, and as the meteor had dropped 40 miles ($70 \dot{-} 30$) during its flight, the length of its path through the atmosphere is $\sqrt{(42^2 + 40^2)} = 58$ miles. If the time of flight is estimated to be 3 seconds, the velocity of the meteor is $58/3 = 19{\cdot}3$ miles a second.

The computations have been carried out on the assumption that the observers are on a plane, no allowance being made for the curvature of the earth. This latter has very little effect unless the path of the meteor is very long, and in many cases it can be completely ignored.

The above method shows the principle used, but the actual solution of the problem is not always so simple as it seems. One great difficulty occurs in the possibility that an observer may miss a portion of the path of the meteor, and when this takes place fictitious results may be obtained for its flight. In these cases there is always a discrepancy between the heights calculated from each observer's base line, and in many instances this discrepancy is considerable. For instance, the use of the base line from Epsom to Hastings may give a height of 75 miles, and the Epsom-Maidstone base may give a height of 65 miles. Of course, the mean of these could be used, but this is a very unsatisfactory way of overcoming the difficulty, because the discrepancy shows that the meteor did not actually commence its flight over Epsom, and a considerable amount of alteration in the positions of its path is required to adjust the differences in the results.

The writer showed how this trouble could be overcome to a large extent, in *The Journal of the British Astronomical Association*, 46, 8, 1936, June, and this method is now used in this country for computing the real paths of meteors. Unfortunately the method is too abstruse to be repeated in a popular work of this kind, and readers must be content with the above outline of the main principles involved in the computation of a meteor's path.

APPENDIX II

The Size of a Meteor

Various methods for estimating the masses of meteors have been developed, but there are certain assumptions which are necessary in all of these, and these assumptions may be far from true. For this reason strict accuracy in the determination of the masses of meteors of different magnitudes must not be expected, but the results can be accepted as approximately correct in some cases. The following method has certain points to commend it and does not require any abstruse calculations nor anything beyond an elementary knowledge of physics.

First of all it is necessary to find the energy emitted to the earth by the radiation of a star of magnitude m, and this can be done by comparing its emission of energy with that of the sun.

It is known that the sun, a star of magnitude $-26{\cdot}7$, sends to each square cm. of the earth's surface about $1{\cdot}4 \times 10^6$ ergs per second—a result based on experimental evidence. By (63), in Chapter X, a star of magnitude 0 should send $2{\cdot}512^{-26.7}$ of this amount, that is $2{\cdot}1 \times 10^{-11}$ of the sun's energy, which implies $2{\cdot}9 \times 10^{-6}$ ergs per second.

Imagine that we are dealing in the first instance with a fairly bright meteor of magnitude 0 which is at an average distance d cm. from an observer during its flight. It is radiating over the surface of a sphere whose area is $4\pi d^2$, and assuming that it is radiating like a star of magnitude 0, its total energy is $4\pi d^2$ that of the star, or $3{\cdot}6\, d^2 \times 10^{-4}$ ergs per second.

In the cases where the magnitude of the meteor is m the total energy is

$$3{\cdot}6 \times d^2 \times 10^{-4} \times 2{\cdot}512^{-m} \text{ ergs per second.}$$

Let the velocity of the meteor be v cm/sec and let its mass be M gm. Its kinetic energy is $\frac{1}{2}Mv^2$ ergs, and if its time of flight is t sec., its kinetic energy per second is $\frac{1}{2}Mv^2/t$ ergs. Equating this with the total energy derived above,

$$\tfrac{1}{2}Mv^2/t = 3{\cdot}6d^2 \times 10^{-4} \times 2{\cdot}512^{-m}, \text{ or}$$

$$M = 7{\cdot}2\, t\,\frac{d^2}{v^2}\, 10^{-4} \times 2{\cdot}512_{-m}.$$

Certain figures which can be taken as correct for the average meteor can be substituted in the above equation. The velocity of the meteor can be assumed about 20 miles per second, or say 3×10^6 cm/sec. Its distance from the observer, taking the mean of the distances at the beginning and end of its course, can be taken as about 60 miles or 10^7 cm., and t, its time

of flight, as 3 seconds. The average meteor visible to the naked eye has a magnitude 3 or 4, but we shall deal first of all with a brighter meteor—one with magnitude 1. Substituting these figures in the above, it is found that

$$M = 10^{-2} \text{ gm.}$$

If the magnitude of the meteor is 2 the mass is $2{\cdot}512^{-1}$ of this, or 4×10^{-3} gm., and if its magnitude is 3 the mass is $1{\cdot}6 \times 10^{-3}$ gm.

To find the size of the meteor an assumption must be made about its density. If it is iron with density about 7·9, and if its radius is r cm., assuming it is spherical, then in the case of a third magnitude meteor,

$$\frac{4}{3}\pi r^3 \times 7{\cdot}9 = 1{\cdot}6 \times 10^{-3}, \text{ from which}$$

$$r = 0{\cdot}037 \text{ cm.}$$

If the meteor is faint—about magnitude 5—the right side of the formula is $1{\cdot}6 \times 10^{-3} \times 2{\cdot}512^{-2}$, because this meteor is two magnitudes greater than the meteor of magnitude 3, whose mass is $1{\cdot}6 \times 10^{-3}$ gm., and hence it is necessary to multiply the above radius by the cube root of $2{\cdot}512^{-2}$, that is, by the cube root of 0·16, which is 0·54. Hence the radius of the fifth magnitude meteor is 0·02 cm. approximately, or its diameter is about one-sixtieth of an inch.

APPENDIX III

Velocity of Escape

Each of the heavenly bodies has a certain *velocity of escape* for objects projected from their surfaces, this term being applied to that velocity which would carry them off from the gravitational attraction of the body. In the case of the earth this velocity is almost 7 miles per second, which implies that if a body is shot from the earth with this velocity it will move off into space, not returning again to the earth, or it may be captured by some other heavenly body. Conversely, a body falling from an infinite distance towards the earth, solely through the earth's gravitational pull, the earth and the body being regarded as an isolated system, would strike the surface of the earth with a velocity of nearly 7 miles per second.

To find the velocity of escape in the case of any other body, let M be its mass and d its diameter, those of the earth being taken as the unit in each case. Then, if V is the velocity of escape,

$$V = 7\sqrt{(M/d)} \text{ miles per second.}$$

Take the case of Jupiter in which $M = 318{\cdot}4$ and $d = 11{\cdot}2$

$$V = 7\sqrt{(318{\cdot}4/11{\cdot}2)} = 7\sqrt{28{\cdot}4} = 37 \text{ miles per second.}$$

APPENDIX IV

Finding the Age of the Moon

The method for arriving at the simple formula given below would require too much space if dealt with fully. Those who wish to understand the reasons for the method should consult *The Journal of the British Astronomical Association*, 51, 9, 1941 October, pp. 313–18, where the writer has given a full investigation of the subject. It should be said that it does not profess to give absolutely accurate results and the ages of the moon obtained by the formula and tables may be in error by as much as two days, but this will often be close enough for practical purposes.

In the first instance the moon's age will be found for January 1 in any year and then it will be shown how it can be derived on any date in the same year. The moon's age on January 1 is known as the *epact* and will be denoted by E.

Let y denote the year and the subscript r the remainder obtained when y is divided by a number. Then the age of the moon on January 1 in any year y is given by

$$E = \left(\frac{11\left(\frac{y}{19}\right)_r}{30}\right)_r.$$

For the present century deduct 1 from the above value of E.

To apply this formula, take as an example the age of the moon on January 1, 1832.

$$\left(\frac{y}{19}\right)_r = 8, \quad \left(\frac{11 \times 8}{30}\right)_r = 28 \text{ days} = E.$$

Dividing 1832 by 19, the remainder is 8. We are not concerned with the quotient. Multiplying 8 by 11 and dividing the result by 30, the remainder is 28, which is the moon's age, or the epact, on January 1, 1832.

To find the moon's age on any other date in the same year it is necessary to find the number of days from January 1 to this date, add

this to the moon's age on January 1, and divide by 29·53, the remainder being the moon's age. To simplify the computations the two tables given below have been compiled, and from these the moon's age can be deduced.

TABLE I

Month		*Add to epact increased by date*	*Month*		*Add to epact increased by date*
January..	..	—1	July ..	..	180
February	..	30	August ..	..	211
March ..	..	58	September	..	242
April ..	..	89	October	..	272
May ..	..	119	November	..	303
June ..	..	150	December	..	333

TABLE II

29·53 × 1	..	29·5	..	29·53 × 7	..	206·7
2	..	59·1	..	8	..	236·2
3	..	88·6	..	9	..	265·8
4	..	118·1	..	10	..	295·3
5	..	147·7	..	11	..	324·8
6	..	177·2	..	12	..	354·4

In Table I the numbers from March to December must be increased by 1 in the case of a leap year. The use of the tables will be shown from a few examples.

EXAMPLE 1

Find the age of the moon on April 10, 1832.

The epact is computed by the formula above and this gives 28 days, as already shown. Since 1832 is a leap year we must add 90—not 89—to the epact, and then to this the date, 10. This gives 128, and from Table II it is seen that the nearest number to this, less than 128, is 118·1. Deducting 118 from 128 the result is 10 days, which is the moon's age on April 10, 1832.

EXAMPLE 2

Find the age of the moon on 1999 August 11.

$$\left(\frac{1999}{19}\right)_r = 4, \quad \left(\frac{11 \times 4}{30}\right)_r = 14.$$

As we are now dealing with the present century, deduct 1 from 14 and the epact is 13.

As 1999 is not a leap year, add 211 for August (Table I), and to this add 11 for the date in August. From the result, which is 235, deduct 206·7 given in Table II, this being the nearest number, less than 235, to 235. This gives the moon's age as 28·3 days. There is a total eclipse of the sun on this date (see p. 139).

Example 3

Find the age of the moon on 1940 January 31, and also on April 6.

$$\left(\frac{1940}{19}\right)_r = 2, \quad \left(\frac{11 \times 2}{30}\right)_r = 22.$$

Hence the epact is $22 - 1 = 21$ days, 1 being deducted for the present century.

For January 31 add to the epact -1, and 31. The result is 51, and Table II shows that 29·5 must be deducted from this, leaving 21·5 days as the moon's age.

For April 6 add 90 (remembering that we are dealing with a leap year which affects the figures after February) and 6. The result is 117, and deducting 88·6 from this the result is 28·4 days.

As already remarked, it is possible for errors of about two days to occur in the calculations by this method, but generally they are much less than this and an error of a day is as much as will usually occur.

The Julian Day

In some lunar problems, such as calculating the dates of eclipses from the cycle referred to on p. 141, it is more convenient to use the Julian day than the usual method of reckoning dates. The Julian day is the number of days that have elapsed since noon on January 1, 4713 B.C. This system was devised by Joseph Scaliger (1540–1609), whose father's Christian name was Julian, hence he called it the Julian system. It is used by variable star observers, and it is very easy to reduce Calendar dates to Julian dates or *vice versa* by means of the tables provided in the *Nautical Almanac*. Those who use these tables (see pp. 1–5 and 534–5 in the *Nautical Almanac* for 1947) must remember that the Julian day begins at Greenwich noon.

ANSWERS

CHAPTER I

1. (a) 17°. (b) 77° 30′. (c) 180°.
2. (a) 46° 57′. (b) 44° 33′. (c) 177° 37′.
3. 39° 10′·5 W.
4. 370 nautical miles.
5. 4327 ft.
6. Nearly 12½ minutes.

CHAPTER II

1. 38° 42′.
2. 37° at the equinoxes. 60° 27′; 13° 33′ at the solstices.
3. 68° N.
4. 47° 15′.
5. 61° 06′ N.
6. 80° 06′. 9° 54′.
7. 56° 33′. On the shortest day of the year the sun's declination is − 23° 27′. Apply (11).
8. $8^h\ 15^m\ 12^s.78$.
9. $1^h\ 32^m$ 12.78.
10. $1^h\ 26^m\ 18^s$.
11. On these dates the sun should be placed at either point of intersection of the equator and the ecliptic. Measure the arc from the meridian to the sun along the equator.

CHAPTER III

1. $7^h\ 31^m\ 06^s$.
2. $7^h\ 54^m\ 34^s$.
3. $18^h\ 53^m$; $5^h\ 07^m$ approximately. 107° 03′ E. and 107° 03′ W.
4. 9° 24′ W. 8°. The latitude and declination are both south and hence both ϕ and δ can be taken with the + sign. Use (16) to find z and then (18) to find A. Because $h = 10^h\ 51^m$, which is less than 12^h, A must be

west. Notice that in the southern hemisphere A is reckoned east and west from the *south*.

5. 74° 49′ E. Take ϕ and δ with the + sign and apply (22). At sunrise the sun's azimuth must be E. and is reckoned from the south in this case.

6. 16° 13′. In (22) A and δ are known; solve for ϕ.

7. 58° 28′. In (21) substitute 45° for h and − 23° 27′ for δ.

8. ~~38° 44′.~~ 57° 16′ and latitudes north of this.

9. $15^h\ 20^m$ and $8^h\ 40^m$, ignoring the seconds of time. 22° 05′ E. and W.

10. $20^h\ 40^m$ and $3^h\ 20^m$. 157° 55′ E. and W.

11. 113° 02′. 6782 nautical miles. See Ex. 10 for the method of solution.

CHAPTER IV

1. 20 minutes.
2. (a) $14^h\ 52^m\ 26^s.233$. (b) $17^h\ 56^m\ 06^s.295$. (c) $3^h\ 13^m$ 28.695.
3. 52° S.
4. $11^m\ 09^s$ before noon.
5. Cos h is found to be numerically greater than 1 which is impossible. Hence the physical interpretation is that the sun neither rises nor sets at the time, remaining above the horizon all the time.
6. From + 10° to + 23½°.
7. About $1^h\ 22^m$ after sunset and before sunrise.
8. 69° 43′. 147° 44′ W.

CHAPTER V

1. 60° 32′ 12″.92.
2. 51° 28′ 38″.34. 58° 50′ 57″.78.
3. (a) 14·28 nautical miles. (b) 15·43 nautical miles.
4. (a) 15′ 03″. (b) 16′ 16″. Find d in each case. The dip is 60d″.
5. (1) $2^h\ 41^m\ 48^s$, $21^h\ 25^m\ 24^s$. (b) $3^h\ 25^m\ 12^s$, $20^h\ 42^m\ 00^s$. (c) $3^h\ 54^m\ 40^s$, $20^h\ 12^m\ 32^s$.
6. (a) 4 minutes earlier. (b) 4 minutes later. (c) 5 minutes later.
7. 67° 10′.
8. 39° 34′ 39″.85.
9. 39° 34′ 39″.88.
10. 52° 21′ E. and W. 125° 20′ E. and W.
11. $8^h\ 9^m$; $15^h\ 48^m$; $8^h\ 11^m$; $15^h\ 57^m$ to the nearest minute.

CHAPTER VI

1. $25^\circ\ 00'\ 23''.44$. Use (33) to find R which has not been included in the observed zenith distance. Then use (39).
2. 92,574,000 miles.
3. 221,634 miles. 2160 miles.
4. $56^\circ\ 58'\ 03''.56$.
5. 137,148,000 miles. 7580 miles.
6. 526,471,000 miles.
7. 252,315 and 225,953 miles respectively.
8. $16'\ 25''.90$ and $14'\ 42''.89$.
9. $0''.025$ nearly.
10. About 91,341,000 miles. 865,752 miles.
11. About 1353 miles.
12. 139 miles.
13. That the diameter of the crater is nearly at right angles to the line drawn from the observer to the crater. The assumption is justified because the crater is near the centre of the moon's disc.

CHAPTER VII

1. R.A. $3^h\ 20^m\ 01^s.66$. Dec. $+\ 49^\circ\ 38'\ 57''.7$.
2. R.A. $7^h\ 19^m\ 16^s.66$. Dec. $+\ 22^\circ\ 11'\ 49''.0$.

CHAPTER VIII

1. 1·5170 astronomical units. 154,941,000 and 128,483,000 miles.
2. 3·554 astronomical units.
3. 1/1047.
4. 17·9422 astronomical units.
5. 3283 and 54·6 million miles respectively. Nearly $25\frac{1}{2}$ miles per second.
6. If a is infinite $v = 18{\cdot}49\ \sqrt{1{\cdot}6666} = 23{\cdot}87$ miles per second. When the comet is at aphelion r is large and is only a little less than $2a$ and hence $2/r$ ~~in (58) is small.~~ For this reason $1/a$ cannot be ignored as its omission would make a relatively large difference in the right hand side of (58).
7. About 84 years.
8. $1{\cdot}0040553$ years $= 366{\cdot}73$ days.
9. $0^\circ.003979$ or $14''.32$.
10. 0.000051.

CHAPTER IX

1. 3·25.
2. 6·73.
3. 0.7.
4. —0·25.
5. 2·44.
6. 8·63.
7. 1·05.
8. 2·25.
9. $12{\cdot}25 \times 10^{10}$.
10. The sun's distance from the earth is 500 light seconds or 0·00001585 light-year. Substitute this value for L in (65) and the result is 4.85.
11. 14·9 times the sun's diameter. Apply (66) and then substitute m in (67).
12. Log 12·6 = 1·1 and the curve shows that $m_a = -3$. From (66) log $p = -4{\cdot}5$, or $p = 0''.0000316$. Hence the distance is 31,650 parsecs or about 103,000 light-years.
13. 5·46. Nearly 11×10^{33}.
14. Nearly 2.

PART II

A BRIEF EXPOSITION OF RELATIVITY

PREFACE

In Part II on Relativity I have attempted to avoid two extremes—abstruseness on the one hand and over-simplicity on the other. This part is intended for readers who have not a mathematical equipment, but it is not entirely devoid of mathematics. Popular explanations of Relativity without any mathematics have not always been a success, and too often they have left the reader with confused and erroneous impressions. Readers are expected to have an elementary knowledge of geometry, algebraic transformations, and also of mechanics, and if they possess this qualification they will find no difficulty in understanding the principles developed in the following chapters.

Examples showing the application of the principles which are explained in the text are given at the end of the chapters, and these should be worked by those who are anxious to gain an intelligent grasp of the subject. As several are worked in full to explain the use of the formulae, the remaining problems should present no serious difficulty.

Although this part is intended to be a popular exposition of a subject which is not very simple, readers must not imagine that it can be read and understood without serious concentration. It is recommended that those who have previously read little or nothing on the subject should study each chapter carefully and not try to hurry through the text in an hour or two. To understand Relativity it is necessary to live in it and to readjust our ideas—and this is not always a simple process for those who have been accustomed to the usual Newtonian mechanics. A special instance of this readjustment is found in the case of the length of an object, which we usually assume to be something absolute and an intrinsic property of the body. As explained in Chapter II, we must discard this view, and to assist in the process the analogy of "weight" explained in Chapter VI will prove profitable. If readers can readjust their conceptions on this particular portion of the problem they will find that the other questions relating to time, velocity and mass will fall into their proper place and will be easy to understand.

The Generalized Theory of Relativity is dealt with towards the end and should not present any special difficulties if the earlier portion

of the work is understood. A very brief account of the crucial tests which gave the verdict in favour of Einstein is given, but the mathematics of the subject cannot be dealt with in an elementary work of this description. The Appendix contains a certain number of formulae and the method for developing them, which may be of interest to a few readers, but they are not essential and need not be read by the amateur mathematician.

It must not be imagined that everything will be known about Relativity by reading this exposition. It makes no pretence to do more than supply a mere outline of the subject, but it is hoped that it will make it easier for those who read it to study Relativity much more fully in more advanced works. A list of books on the subject is given at the end, and the works are arranged in order of increasing difficulty so that readers will know where to start if they wish to continue their study of Relativity. If the present treatment renders the pursuit of the subject easier, the author will feel that his labours have not been in vain.

Chapter I

INTRODUCTORY REMARKS

It is trite to remark that most of the terms which we use are relative, though very often we are unaware of the fact. Take, for instance, the expressions "up" and "down". When we use these words we think we understand exactly what they mean and no doubt we generally do, but perhaps we sometimes forget that unless we define the object with reference to which anything is up or down the terms are meaningless. When we speak of anything going up in this country we imply that it is moving approximately in a line drawn from the centre of the earth to our position on the earth's surface, and in a direction away from the earth's centre. A New Zealander implies the same thing when he is describing the meaning of the word "up" in his own country, but with reference to the earth's centre, a distant star, and also to some of our closer neighbours—the planets—the two directions are nearly opposite. This simple illustration shows us that if we wish to be very accurate in our descriptions of directions we must define our terms more clearly than we have been accustomed to do.

The same principle must also be recognized in dealing with quantitative results. Very often people speak of an object as "big" or "small", but obviously these words have little meaning unless we know what is our standard of comparison. A minor planet is big in comparison with the size of a house but very small in comparison with the size of the earth. The earth itself is considered a small planet when it is compared with Jupiter, and Jupiter is very small when we place it beside the sun. Actually Jupiter is more than 1300 times the volume of the earth, and the sun is about 1000 times the volume of Jupiter, but the sun itself is very small in comparison with some of the giant stars such as Betelgeuse, Antares and others. The last star is more than 100 million times the size of the sun. In spite of the enormous size of Antares it dwindles into insignificance when compared with the size of the Galaxy.

At the other end of the scale we speak of some things being very small, such as bacilli, viruses, etc., but what is our standard of comparison? If it is some of the ordinary forms of life which we find in our ponds, such as the amoeba, the paramecium, the rotifer, etc., and which present many interesting features when we look at them with an ordinary microscope, it may be admitted that bacilli are small. If, however, we compared bacilli with atoms we should be obliged to admit

that they were very large and that the atom was extremely small. But the atom is no longer regarded as small since the discovery of the electron, and we now know that the atom occupies an enormous space in comparison with that occupied by the electron. It is unnecessary to multiply instances, and we must return to our starting point and repeat that a great many of our terms are relative and that they are meaningless unless we adopt some standard of reference.

When we deal with motion and velocity there are many pitfalls unless we are careful to define our terms very carefully. If we are on a boat which is moving with a speed of 15 knots we think we know exactly what our velocity is, and we should be prepared without hesitation to assert that it is 15 nautical miles an hour. But what is our reference point? If we are prepared to take some landmark, a buoy or a rock, and to say that relative to this we are moving at a certain speed, no serious objection would be raised to our statement. Now, however, consider some of the other motions in which we are taking part and of which we may be unaware when we carelessly speak of our speed. With reference to the centre of the earth we are moving with a velocity of more than 1000 miles an hour, if our boat is in equatorial regions. If it is in latitude 30° this speed is about 870 miles an hour, and in latitude 60° it is only half of what it was at the equator. These speeds are in addition to the speed of the boat—15 knots—and may be in the same or in opposite directions to the boat's motion or in intermediate positions. Then, if we want to be more accurate still and to determine our speed with reference to a body outside the earth—say the sun—we must take into consideration another motion, that is, the orbital motion of the earth, which is nearly 18½ miles a second, as it revolves round the sun. Even this does not exhaust all the motions that we experience, because the sun itself is moving in the local star cloud, which, in turn, is moving round the centre of the Galaxy, completing a revolution in about 220 million years. If, therefore, we thought it quite sufficient to confine our calculations to that comparatively small portion of the universe known as the Galaxy, which consists of about 100,000 million stars, we should attempt to determine our velocity with reference to the centre of this system. What the centre of the Galaxy is doing need not concern us at the moment. We believe that it is moving away from the centres of other galactic systems but this is of little interest for us at present.

So far we have seen that motion is relative and we can find our relative speed without much difficulty when we are dealing with terrestrial objects and standards. But now suppose we are dissatisfied with this limited attainment and start out to discover where we are going or towards which galaxy we are moving, what procedure should we adopt? A simple illustration from the case of a boat will assist us in answering this question.

While the boat previously considered is moving with a velocity of

1520 feet a minute relative to a buoy, imagine that a passenger paces the deck with the speed of 240 feet a minute, relative to a mark on the deck; it is not difficult to find his speed relative to the buoy, and this will depend on the direction in which he is walking. If he is moving in the same direction as the boat his speed relative to the buoy is 1760 feet a minute, and if he is moving in the opposite direction it is 1280 feet a minute. If he is moving across the deck at right angles to the boat's direction of motion, his speed relative to the buoy is found from the simple principle of the parallelogram of velocities and is just under 1539 feet a minute. We have no hesitation in applying the ordinary elementary principles that we learned at school to obtain these figures, but, as will appear later, they are not strictly correct, though the reader may accept them as correct for the present. Later on it will be shown that they are based on a fundamental fallacy.

It seems fairly obvious that it might be possible to detect the motion of the earth through the ether because, assuming that the earth is moving through the ether, this is the same thing as if the ether is streaming past the earth. If you are rushing through the air, relatively it is the same as if the air were rushing past you, and an object projected by you in the direction of your motion will not have the same speed as it has when projected in the opposite direction. Speed in these cases is measured with reference to some mark on the ground. In the same way the velocity of light should be slower when it is moving against the ether stream than when it is moving with it. We shall return to this point in the next chapter.

Chapter II

HOW EINSTEIN'S THEORY AROSE

An Experiment with Two Boats on a River

Figure II (1) represents a river, the thick lines being the banks, and the arrow showing the direction in which the stream is flowing with a uniform velocity of 8 feet a second. Two men set out from a point *P*, each equipped with a motor-boat capable of moving 10 feet a second in still water, one going down stream and the other across stream. It will be

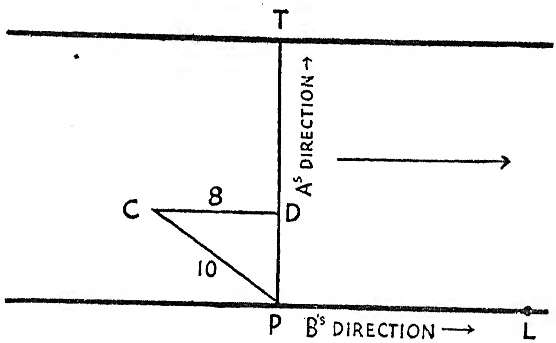

Fig. II (1)
A simple experiment with boats crossing a stream and moving up and down stream.

assumed that there is no wind and that the speed of each boat remains exactly the same all the time. The other bank is 180 feet distant from *P*, measured at right angles to the direction of the stream, and *A* decides to take his boat across to *T*, directly opposite *P*, and back again to *P*. *B* decides to take his boat to *L*, which is 180 feet from *P*, measured down stream, and to return to *P*, and a discussion arises regarding the time that each boat will require. Most readers know the answer without calculations, but perhaps a few are doubtful, and for these we propose dealing with the problem at length.

A will be very far out in his reckoning if he sets his course straight for *T* because the stream will carry him down a considerable distance in the direction of the arrow, and instead of finding himself at *T* he would reach the other bank a long way down stream. If he knows how to steer

his boat correctly he will set his course along the direction *PC*, which can be calculated as follows.

For every 10 feet that the boat moves along *PC* the stream will carry it 8 feet down stream, so it is necessary to arrange the direction of *PC* in such a manner that if *PC* is 10 feet and *CD* is 8 feet, the point *D* will lie exactly on the line joining *P* and *T*. It must not be imagined that *A* ever reaches the point *C*; the stream is making the boat drift every instant, so when the prow is set parallel to *PC* the actual course of the boat will be in the direction *PT* and *A* will reach the opposite bank exactly where he intended to go. (The problem, as the reader will see, merely involves the simple application of the parallelogram of velocities.)

What time will *A* require to accomplish his journey? To answer this question it is necessary to find out what his speed across the river is. This speed is obviously proportional to *PD*, on the same scale on which *PC* and *CD* represent the velocities of the boat and the stream respectively. Since the angle *PDC* is a right angle, it follows that

$$PD^2 = PC^2 - CD^2 = 100 - 64 = 36$$
$$\text{Hence } PD = 6.$$

The actual speed of *A*'s boat across the river is, therefore, 6 feet a second, and he will require 30 seconds to cross the stream. The return journey will occupy exactly the same time, provided *A* remembers how to steer his boat properly. If he steers it so that the prow is not pointed sufficiently far up stream he will find himself somewhere between *P* and *L* when he reaches the bank. If he points the prow too far up stream he will reach the bank above *P* and will then be obliged to go down stream to reach his goal. This will involve wasting time and he will have failed to do the trip in the minimum time.

How long will *B* take to do the double journey, down stream to *L* and back again to *P*?

Down stream *B* is moving with a speed of 10 feet a second relative to the water, and the stream is carrying him 8 feet a second relative to the bank. Hence his speed relative to the bank is 18 feet a second and he will require 10 seconds to go 180 feet down stream. On his return journey he is still moving with a speed of 10 feet a second relative to the water but the stream is carrying him back with a speed of 8 feet a second relative to the bank, so that his speed relative to the bank is only 2 feet a second. Hence he will require 90 seconds to do the journey of 180 feet up the stream, the total time to make the double journey being 100 seconds. The ratio of the times required to do the double journey along and transverse to the river is 100/60 or 5/3. We can generalise from this case and conclude that the time to cross and recross is always shorter than that required to go up and down stream by the same distance.

Instead of taking the velocity of the boat to be 10 feet a second and

that of the stream 8 feet a second, we shall denote the speed of the boat by c and that of the stream by v. In addition, the width of the river will be denoted by d instead of 180 feet, and by referring to the above example the following expressions* will be obvious and can be checked by taking a number of cases:

A's speed across the river $\sqrt{(c^2 - v^2)}$
B's speed down stream $c + v$
B's speed up stream $c - v$

Time required by A to cross and recross	$2d/\sqrt{(c^2 - v^2)}$
Time required by B to go down stream	$d/(c + v)$
Time required by B to go up stream	$d/(c - v)$
Time required by B to perform the double journey	$2cd/(c^2 - v^2)$

Suppose we want to find the ratio between the times taken by A and B to perform the journey there and back, we divide $2d/\sqrt{(c^2 - v^2)}$ by $2cd/(c^2 - v^2)$ and obtain for the required ratio $\sqrt{(c^2 - v^2)}/c$, which is independent of the distance d. In our example c is 10 and v is 8, and hence the ratio is $\sqrt{(10^2 - 8^2)}/10$, or 3/5, which is the same as that obtained previously.

We can imagine a third party coming along and offering to tell A and B what is the speed of the river if they will supply him with the following information: (1) the ratio of the times that each requires to do the double journey; (2) the speed of each boat (which is supposed to be the same). On informing him that the ratio is 3/5 and that the speed of each boat is 10 feet a second, the equation $\sqrt{(c^2 - v^2)}/c = 3/5$ will provide the answer. By making $c = 10$ the equation then becomes

$$\sqrt{(100 - v^2)} = 10 \times \tfrac{3}{5} = 6, \text{ from which } v = 8 \text{ feet a second.}$$

The Michelson-Morley Experiment

We shall now give a very brief description of an important experiment which was made first of all by Michelson in 1881 and afterwards repeated by Michelson and Morley with the aid of more refined apparatus in 1887. The object of this experiment was to detect the motion of the earth through the ether by the effect on the velocity of light. The principle of this experiment is explained very simply in Fig. II (2), in which the earth and the apparatus are supposed to be moving through the ether in the direction CA, and from the point of view of an observer on the

* General proofs are not always given in the text, as they may deter some readers, but they will be found in the Appendix.

earth the ether is streaming past him in the opposite direction, that is from *A* to *C*. Imagine that *AB* and *AC* are measured carefully and are exactly the same length, and that rays of light are dispatched at the same instant from *A*, one along *AB* and the other along *AC*. In addition, suppose that mirrors at *B* and *C* reflect the two beams back to *A*, which beam will arrive first?

A detailed description of the apparatus is outside our scope and

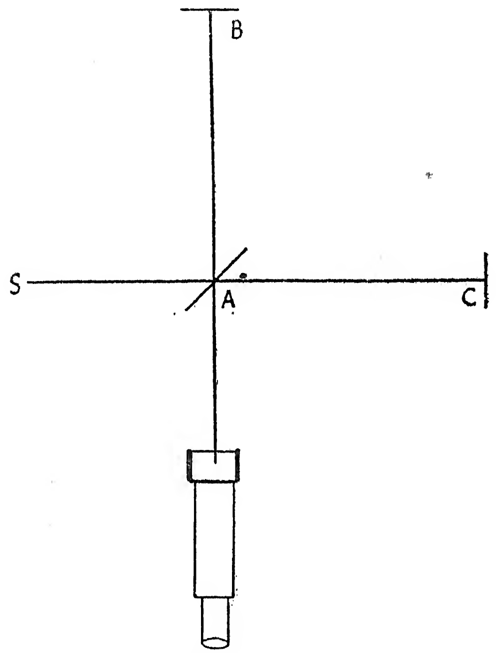

FIG. II (2)
The apparatus for the Michelson-Morley experiment.

readers must consult text-books on physics for a full explanation. It may be pointed out, however, that the mirror at *A* was half silvered, one portion allowing the beam from a source *S* to proceed straight to *C* and to be reflected back again to *A* by which it was reflected into a telescope. The mirror *A* was tilted at an angle of 45° to the direction *AC*, so that its silvered portion reflected the beam to *B*, from which it was reflected back to A and passed into the telescope. By means of the interference fringes it was possible to detect if there was any difference between the times of arrival of the rays at the telescope.

The principle is the same as that of the men in the boats. The velocity of light corresponds to that of either boat, and the velocity of the

earth through the ether corresponds to that of the river. The reader will immediately conclude that the answer to the question, "Which beam will arrive first at the telescope?" is simple. From the analogy of the boats he will say that the beam which was sent transverse to the direction of the earth's motion, AB in the present case, would arrive first. The remarkable thing was that both beams arrived at precisely the same instant, and on repeating the experiment the result was always the same.* It might be suggested that, considering the different motions of the sun and also the orbital motion of the earth, previously referred to, the resultant velocity of the earth relative to the ether happened to be zero at the time of the experiment. This explanation was shown to be untenable by repeating the experiment six months later, when the earth's orbital motion was in a different direction, the result being the same as before. Another suggestion was that the earth dragged the ether with it, in which case no difference in the times of arrival of the beams would be expected. This hypothesis is quite invalid, and no doubt many readers will remember Sir Oliver Lodge's experiments with rotating discs to detect any drag of the ether, all of which gave negative results. In addition, the astronomer is unable to allow such an ether drag because it would vitiate his explanation of the well-known phenomenon of aberration.

The results of this famous experiment exercised a profound influence on scientific and philosophic thought, and it almost seemed that the whole edifice of physical conceptions was crumbling and was destined to fall in ruins. The experiment showed that the earth was not moving, but the astronomer *knew* that it was moving, so the world about which we thought we knew so much was one thing to the physicist and something different to the astronomer. Was there any possibility of reconciling views which appeared contradictory? We have spoken of the ether and of the attempt to detect motion through it, but it is irrelevant for our purpose whether there is such a thing as the ether of space. It is unnecessary to postulate some of the extraordinary properties of the ether which the older physicists assumed, and indeed it is unnecessary to postulate its existence at all, though it will do no harm to assume that it is there. We are reminded of the conversation said to have taken place between Laplace and Napoleon. Napoleon asked Laplace where the Deity came in his *System of the World*, and Laplace replied, "I have no need of that hypothesis." The reply has sometimes been misconstrued and taken to imply that Laplace meant he simply assumed the existence of God and that there was no need to form any hypothesis about the matter. What he really meant was that his scheme did not require the hand of God continually regulating the movements of the heavenly bodies (we are reminded of the theory, before the days of Laplace, that angels pushed the planets along) and that mechanical principles were sufficient. He did not imply that God did not exist, nor did he imply that

* As will be seen later, there is an important exception to this.

He did, but He was just irrelevant for the matter under consideration. On the whole, however, it will assist the reader if he assumes that there is an ether, but he need not concern himself with its properties. The restricted Principle of Relativity tells us that *it is impossible by any experiment to detect uniform motion relative to the ether.*

Reverting to the failure to detect motion through the ether, men of science have now come to the conclusion that the universe which they once believed to be independent of those who perceived it can no longer be regarded in this way. In fact everything that we see assumes a form and content determined by its relation to the observer, and the external world of matter situated in space and time is really all things to all men. There is no meaning in absolute motion. All motion is relative and it depends on our own way of thinking. Let us see how all this is verified by returning to the experiment with the boats.

Certain Implications of the Michelson-Morley Experiment

We must now introduce some ideal or perhaps hypothetical conditions into consideration, but these will not detract from the validity of the argument. First of all we shall take a large lake instead of a river, and imagine that its shores are invisible to A and B. We shall postulate a surface current with a speed of 8 feet a second, but, as no landmarks are visible, A and B will not be aware of this current, and if they shut off their engines they will imagine that their boats are stationary. It will be also necessary to assume that there are rocks or some other obstructions under the water, by means of which A and B can measure distances in the direction of the current and at right angles to it. Having measured these there is no reason why they should not again engage in a competition just as they did on the river. The fact that these rocks might lead them to infer the presence of a surface current need not concern us as we are dealing with very ideal conditions in which a lapse of memory may be helpful. Finally, a balloon is moored to a fixed object, say at the bottom of the lake, and in this balloon an observer C is making careful notes of what takes place. (See Fig. II (3.)

A and B sit in their boats, having stopped their engines, and do not notice that they are drifting with the current. If they look at C they will be convinced that he is moving away from them with a speed of 8 feet a second, and even if there are floating objects on the lake, these will not prove that A and B are moving because these objects will have the same speed as the boats. Each boat represents a ray of light in the Michelson-Morley experiment and the stream represents the motion of the ether in a direction opposite to that of the earth's motion, and we shall imagine that the experiment is repeated on the scale of ordinary terrestrial velocities.

Each person is provided with a standard measure—say a foot rule—and these have been carefully compared and have been found to agree. *A* and *B* measure 180 feet in the direction of the current and at right angles to it, and the positions are marked by the rocks just submerged beneath the water. Knowing nothing about the surface current they believe that their speed is 10 feet a second, and hence they estimate that the double journey of 360 feet in each case will occupy 36 seconds.

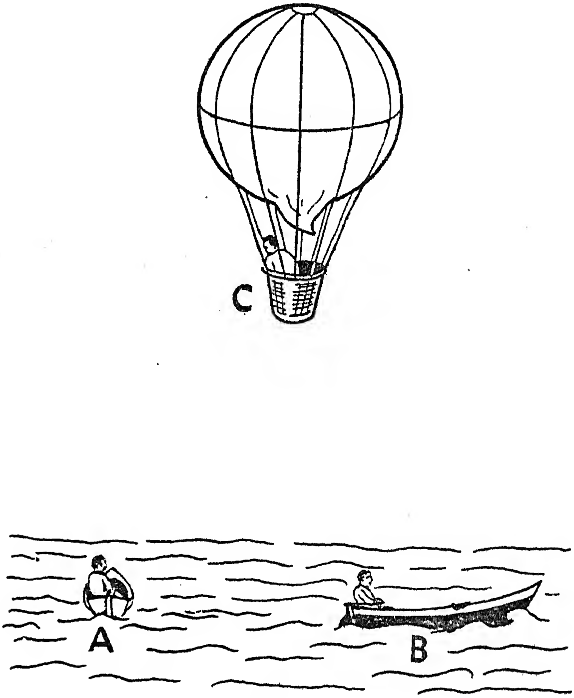

Fig. II (3)
A further experiment with boats in a stream.

It may be objected that this is absurd, because their clocks will show that the times are different, the actual times being as previously given on pp. 169–70. It must be remembered, however, that *we are now performing the Michelson-Morley experiment which shows that the times are the same*, so we must examine the foot-rules and clocks to see if there is any defect in these which will explain the apparent anomaly. The following are the views of each of the three people engaged in this experiment.

As an independent observer *C* knows that *A* is crossing the stream with a speed of 6 feet a second, as we showed previously, and will estimate his time to cross and recross from one rock to the other to be 60 seconds—

not 36 seconds as *A* thinks. *C* will also estimate *B*'s time to go up and down the course to be 60 seconds, because *A* and *B* require exactly the same time for the trip (see p. 172), though *B*, like *A*, is convinced that the time is only 36 seconds. When they all meet to discuss the results of the experiment the following imaginary conversation will show how each of them records his observational results.

C. I have timed your trip very carefully, *A*, and find that it required exactly 60 seconds. I hope you agree with my figures.

A. I am afraid I disagree with you. I timed my trip and found that it required only 36 seconds. I consider your clock does not keep very good time.

B. Did you time me for my trip too?

C. Yes, and I found by my clock that your time also was 60 seconds.

B. I do not agree. I found by my clock that my time was only 36 seconds.

C. I am afraid the trouble lies with your clock, *A*. You say it registered 36 seconds, but in point of fact it should have registered 60 seconds, so it loses very badly. With regard to your statement, *B*, I think I know where your mistake lies, and I should like to explain the matter fully. I observed your movements very carefully and found that while you travelled with a speed of 18 feet a second on the outward trip your speed on the return trip was only 2 feet a second. You believed that the length of each portion of the trip was 180 feet, so that the time required to go there and back would be $180/18 + 180/2 = 100$ seconds, but as I found it was only 60 seconds I must conclude that the length of the half-trip is only $\frac{60}{100} \times 180 = 108$ feet. Your measure is obviously very much in error—in fact it is only 3/5th of a foot.

B. It is difficult to know why you think that my foot rule is so far out. You checked it yourself.

C. True, but under different conditions. It was then held in a certain direction, perpendicular to the direction of a surface stream, of which you seem to be unaware. When it is placed parallel to the direction of the stream its length contracts by the amount that I have just stated. I should like to say, further, that your clock loses at the same rate as *A*'s. You allege that your trip both ways occupied only 36 seconds, but I find that the time was 60 seconds, so your clock loses like *A*'s.

The question now arises, "Who is right?" Each one has equally valid reasons for maintaining his own view on the matter, and how are we to decide which view, if any, is to be accepted? The answer is that all three are right, each from his own point of view. If one world is moving relative to another, the standards of space and time, and, as we shall see later, of mass as well, become different. This may seem a revolutionary idea, or at least it did seem so when the Michelson-Morley experiment upset some of our old-established views, but we are gradually becoming

accustomed to it, and it is no longer regarded as mere speculation. It is based on experimental evidence.

It can be shown by a similar process of reasoning that A and B's ideas about C are the same as those which C formed about A and B, and the following is a summary of the results as judged by each one:

C says that clocks in the world of A and B lose time, registering an interval as only 3/5th of its true value.

C says in addition that a measure placed parallel to the stream records only 3/5th the actual length. If placed at right angles to the stream it measures correctly.

A and B disagree with practically all of this. They maintain that their clocks keep normal time and their foot-rules or any other standards of length remain correct in all positions.

A and B further assert that clocks in C's world lose time and register only 3/5th of the correct interval.

They also say that a measure in C's world placed at right angles to the stream is correct, but parallel to the stream it records only 3/5th of the true length.

The above results are most important and it will be advisable to illustrate them by some examples. For the purpose of numerical illustrations it will be convenient to take the velocity of the boat as the unit and that of the stream as a fraction of this unit. Thus, instead of saying that the velocity of a boat is 10 feet a second we shall call this velocity 1 and that of the stream 4/5, which will be denoted by u. The summary of the views of the different people can then be expressed as follows:

C says that clocks in the world of A and B register an interval which is only $\sqrt{(1-u^2)}$ that of the true interval. A and B assert the same about the clocks in the world of C.

C says that a measure placed parallel to the stream registers only $\sqrt{(1-u^2)}$ of the true length, though placed at right angles to the stream it registers correctly. A and B agree that a measure in C's world, if placed at right angles to the stream, is correct, but say that if it is placed parallel to the stream it registers only $\sqrt{(1-u^2)}$ of the true length. These results can be checked, if the reader so desires, by a number of examples.

There is a well-known elementary principle which is useful for computing approximate results when u is small. This principle is that

$$\sqrt{(1-u^2)} = 1-\tfrac{1}{2}u^2, \text{ and } 1/\sqrt{(1-u^2)} = 1+\tfrac{1}{2}u^2,$$

and the smaller u is the more accurate the results are. Thus, if u is 0·2, the value of $\sqrt{(1-u^2)}$ is 0·9798, and $1-\frac{1}{2}u^2$ is 0·98, the discrepancy being only 0·0002.

Suppose we were asked to find how much the length of the earth's diameter contracted owing to its orbital motion round the sun and also how much a clock loses each day on the earth because of this motion

(about $18\frac{1}{2}$ miles a second), we must define the positions of the observers. The orbital motion is round the sun, so we can imagine that C is at rest relative to the sun and sees the earth carried away from him in the direction of the tangent to the earth's orbit at the time. As the velocity of light is about 186,000 miles a second, which we shall take as the unit, the velocity u of the earth is nearly 0·0001. Substituting this value in the above expression we find that B's clock records $1 - 0{\cdot}000000005$ second according to C, or in other words, it loses 0·000000005 second per second which is 0·000432 second per day.

The change in the length is $\frac{1}{2}u^2$ for each unit of length, and as the earth's diameter is nearly 8,000 miles, the decrease in the length of the diameter which is parallel to the direction of motion is 0·00004 mile, or a little over $2\frac{1}{2}$ inches.

It may seem strange to be told that the earth contracts as a result of its orbital motion, and if we take into consideration the other motions of the earth with reference to some distant star (the motion which it shares with the sun in his journey through space) we should have to allow for other "contractions". The reader must not assume that there is an actual physical contraction, and this can be made clearer by remembering that if the earth is receding from an observer C we can express the statement in a different way by saying that the observer C is receding from the earth. We can scarcely assume that the earth contracts because C is receding from it, though it would not be incorrect to say that its length contracts. There is nothing absurd in this statement because length is not an intrinsic property of a body, though we once believed it was. Length is merely a conception which we associate with every body and which we define as a function of two quantities. These are (1) its length l measured by a *scale at rest with reference to the body*; and (2) the velocity u of the object in *the direction of its length*, relative to our standard of reference.

Times in Different Worlds

We shall now proceed to an important problem which will be solved by means of a special case. It will be shown later how formulae can be derived which are applicable to all cases.

Suppose two people A and B are moving with a velocity 1/3 relative to another person O, the motion being away from O in the direction A to B, Fig. II (4). Let the distance AB be 10 units, which will be taken to be the distance through which light travels in 10 seconds. A and B wish to synchronize their clocks and agree to do so as follows:

A proposes that when his clock registers zero hour he will send a light signal to B, this light signal requiring 10 seconds to reach B. Hence B sets his clock 10 seconds after zero hour and waits for A's light signal.

When he receives it and starts his clock he will have synchronized it with *A*'s clock, and so far the problem seems quite simple. But it will not appear so simple when we have enquired into *O*'s views about this synchronization.

From our previous investigation we know that *O* will judge the distance *AB* to be $10\sqrt{(1-u^2)}$, which is 9·43 since u is 1/3. From *A* to *B* the light ray travels at the rate of 1 unit a second, but as *B* is receding at the rate of 1/3 unit a second, the ray gains on *B* at the rate of 2/3 unit a second. For this reason the time taken by the ray to reach *B* is $9{\cdot}43/\frac{2}{3} = 14{\cdot}14$ seconds. On the return journey from *B* to A the point *A* is advancing to meet the ray and the velocity of the ray relative to *A* is $1\frac{1}{3}$ unit a second, so that the time required for the return journey is 7·07 seconds. The total time required for the ray to travel from *A* to *B* and back is, therefore, 21·21 seconds, and hence *O* says that the outward journey from *A* to *B* occupies 14·14/21·21 = 2/3rd of the whole time.

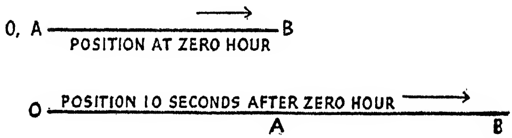

Fig. II (4)

How to derive an expression for the relation between the times in different worlds.

Since *A*'s clock registers 20 seconds for the total time there and back, *O* says that *A*'s clock registers $13\frac{1}{3}$ seconds for the journey from *A* to *B*. Hence *O* says that *A*'s signal to *B* reaches him in $13\frac{1}{3}$ seconds and not in 10 seconds, and when they have synchronized their clocks *O* says that *B*'s clock is $3\frac{1}{3}$ seconds behind *A*'s clock. These figures are easily obtained by multiplying 10, the distance between *A* and *B*, by 1/3, the velocity of *A* and *B* relative to *C*. In all cases the problem can be solved by the simple relation,

$$t_1 - t_2 = us$$

where t_1 and t_2 denote the times of the clocks of *A* and *B* respectively, u is the velocity of *A* and *B* with reference to *O*, and s is the distance between *A* and *B* expressed in the selected unit—the distance through which light travels in one second.

The following problem will illustrate the application of the above formula.

A and *B* are at rest with respect to each other but they are moving relative to *O* with a velocity 7/25. They are separated by a distance of 20 units and have synchronized their clocks. *A* passes *O* at zero hour by

the clocks of both *A* and *O*. What, according to *O*, is the difference between *A*'s clock and *B*'s clock (1) when the direction of motion is *AB*; (2) when the direction of motion is *BA*?

$$(1)\ t_1 - t_2 = 20 \times \frac{7}{25} = 5{\cdot}6$$

$$(2)\ t_2 - t_2 = 20 \times -\frac{7}{25} = -5{\cdot}6$$

In the second case u is negative because *B* is approaching *O*. In the first case *A*'s clock is 5·6 seconds ahead of *B*'s, and in the second case it is 5·6 seconds behind *B*'s, according to *O*.

Suppose that *A* holds a foot-rule parallel to the direction of his motion relative to *O*, what would be *O*'s estimate of the length of the rule?

According to *O* the length of the rule is $\sqrt{(1 - u^2)} = 24/25$ foot. Hence in *O*'s world the length of the foot-rule is over $11\frac{1}{2}$ inches. Comparing this with the case of the earth in its orbital motion round the sun, it will be seen how the change of length increases rapidly with increase in velocity.

Before proceeding to examine a number of other relations in different universes in which motion of one relative to another takes place, something will be said on an exception to the statement that all attempts to measure the velocity of matter with respect to the ether have failed.

Professor Dayton Miller conducted a number of experiments with a refined form of the Michelson-Morley interferometer, and as a result concluded that there are definite effects. He believed that not only is the earth's orbital motion indicated, but in addition, the motion of the solar system through the ether is suggested, the velocity of this motion being about 130 miles a second. In the Michelson-Morley experiment very slight displacements of the fringes were noticed and these were attributed to experimental errors, but Miller thought that they were real effects due to a dragging of the ether by the earth, or to the operation of a modified Fitzgerald contraction.

It is difficult to explain the results of Professor Miller's experiments, and if we accept their validity we must account for many other experiments which showed that there was no effect. It is unfortunate that some satisfactory explanation was not forthcoming before Miller's death some years ago. If his results were correct there would be no object in proceeding with the present work, but we shall accept the results of the Michelson-Morley experiment, as is done amongst practically all physicists, and proceed with our explanations.

The negative result of the Michelson-Morley experiment was explained first of all by Fitzgerald, a Dublin physicist, and afterwards by Larmor and Lorentz. It was suggested that a material body moving

through ether is automatically contracted by a factor $\sqrt{(1 - u^2)}$ in the direction in which the component of velocity is u. If this were true the length l of a body at rest would become $l\sqrt{(1 - u^2)}$, and the experiment would fail to give us any knowledge of the earth's motion through the ether, because the standard with which a distance is measured would contract in the same proportion as the distance itself.

We do not propose dealing with the subject from the point of view of the Fitzgerald contraction as this is liable to mislead the reader. When we say that a body contracts on moving we express the Fitzgerald contraction hypothesis correctly, and we can imagine an actual physical contraction. This, however, is different from the hypothesis of relativity because, as we saw on p. 177, length is not an intrinsic property of a body.

The word "clock" has been frequently used and requires some explanation. It is not implied that observers carry about with them time-measuring instruments exactly like our clocks or watches. A clock is simply a mechanism for measuring time intervals accurately, and may be a pendulum, a water-clock, a sundial, or various other forms of apparatus. Just as we must not speak in relativity about actual physical contractions of bodies in motion, so we must not imagine that a clock's rate is altered by motion. We change our unit of time in such a way that it is merely the time taken by a moving body to cover a selected number of units of length. On referring to the conversation between A, B and C (p. 175) it is obvious that the modification in the definition of length implies also a modification of the unit of time. Instead of l and t in a universe at rest relative to an observer O, we must take $l\sqrt{(1 - u^2)}$ and $t\sqrt{(1 - u^2)}$ when the speed of the universe relative to O is u.

CHAPTER III

RELATION BETWEEN TIME- AND DISTANCE-INTERVAL

WE shall now proceed to derive important relations between time- ar distance-intervals in two worlds which will be denoted by O and Subscripts will be used in the symbols employed for each world: th s_0, t_0, and s_a, t_a refer to space- and time-intervals in the world of O an A respectively, and u will be used throughout to denote the velocity of relative to O or of O relative to A.

Fig. II (5) shows A and B moving with velocity u in the direction arrow, the distance interval AB being s_a as measured by A or B. will be the same for each, as A and B have no motion relative to eac

FIG. II (5)
Derivation of the relation between time- and distance-intervals in different worlds.

other. When A is passing O at zero hour event 1 occurs, and event occurs at B in t_0 seconds after zero hour by O's clock. O says that th distance-interval between the two events is s_0, which is his measure the length of OB (see p. 178). Since A is moving away from O with velocity u, in t_0 seconds A has moved a distance ut_0 away from O. Fror the diagram we see that

$$AB = OB - OA = s_0 - ut_0$$

and hence O says that by his rule the right-hand side of the abov expression is AB.

A measures AB as s_a and O says that this distance is $s_a \sqrt{(1 - u^2)}$ Equating the two expressions for the length of AB according to O,

$$s_0 - ut_0 = s_a \sqrt{(1 - u^2)}$$

from which

$$s_a = (s_0 - ut_0)/\sqrt{(1 - u^2)}.$$

The corresponding expression for s_0 can be found by similar reasoning or it can be written down from symmetry, remembering that when we wish to find s_0 it will be necessary to change the sign of u. Hence

$$s_0 = (s_a + ut_a)/\sqrt{(1 - u^2)}.$$

To find t_a and t_0, let $\sqrt{(1 - u^2)} = k$. Substituting this value for $\sqrt{(1 - u^2)}$ in the two expressions for s_a and s_0, we obtain

$$s_a = (s_0 - ut_0)/k, \text{ or } s_0 - ut_0 = ks_a$$
$$s_0 = (s_a + ut_a)/k, \text{ or } s_a + ut_a = ks_0$$

Multiplying the first of the above equations by k and transposing the terms, we obtain

$$ks_0 = k^2 s_a + kut_0.$$

But $ks_0 = s_a + ut_a$ by the second equation. Hence,

$$k^2 s_a + kut_0 = s_a + ut_a, \text{ from which}$$

$$kut_0 = (1 - k^2)\, s_a + ut_a = u^2 s_a + ut_a, \text{ since } 1 - u^2 = k^2 \text{ or } 1 - k^2 = u^2.$$

Dividing by u we obtain

$$kt_0 = us_a + t_a, \text{ from which}$$
$$t_0 = (t_a + us_a)/\sqrt{(1 - u^2)}.$$

The value of t_a can be found in a similar manner, and the four equations connecting the time- and distance-intervals between the two events are as follows:

$$s_a = (s_0 - ut_0)/\sqrt{(1 - u^2)} \qquad t_a = (t_0 - us_0)/\sqrt{(1 - u^2)}$$
$$s_0 = (s_a + u_a t)/\sqrt{(1 - u^2)} \qquad t_0 = (t_a + us_a)/\sqrt{(1 - u^2)}$$

Although these formulae have been derived by elementary algebraic transformations it is possible that a few readers may find it difficult following the work. If so, they must be prepared to accept them and they will find that their practical application is a simple matter. Two examples follow, and the first of these is solved by the use of the formulae and also by dealing with the problem merely as a particular case. This latter method will show the reader the justification for the formulae which have been derived.

An observer O says that A's world is moving away from him due east with a velocity of 4/5 unit. A's world says that there are two special events and that the second occurs 6 units due east of the first and 10 seconds later. How does O record the time- and space-intervals between the events?

The data are as follows:

$$s_a = 6,\ t_a = 10,\ u = 4/5, \text{ and hence } \sqrt{(1 - u^2)} = 3/5.$$

Hence

$$s_0 = \left(6 + 10 \times \frac{4}{5}\right) \Big/ \frac{3}{5} = 23\tfrac{1}{3}$$
$$t_0 = \left(10 + 6 \times \frac{4}{5}\right) \Big/ \frac{3}{5} = 24\tfrac{2}{3}$$

This problem can be solved by dealing with it as a particular case, and this will be done for the space-interval.

A says that AB is $OB - OA$ and O says that OB is s_0 which is as yet unknown but which will be obtained. A does not agree with O that OB is s_0 and asserts that it is $s_0\sqrt{(1 - u^2)} = 3s_0/5$. In addition, A says that OA is $4t_a/5$, because in A's time t_a he has moved with velocity $u = 4/5$. Hence A says that OA is $\frac{4}{5} \times 10 = 8$. We have seen that A says that AB is $OB - OA$ or $3s_0/5 - 8$, but he also says that AB is 6, because in his world the event occurred 6 units east of A. Hence

$$3\,s_0/5 - 8 = 6, \text{ or } 3s_0 = 70, \text{ from which } s_0 = 23\tfrac{1}{3}.$$

Another observer O' says that A is moving from him with a speed of $4/5$ due west. How does O' record the interval between the events?

The diagram shows that in this case $u = -4/5$. In the first example A's world was moving eastward and the second event occurred 6 units due east of the first. In the second example the second event occurs 6 units due east of the first, but A's world is not moving due east; it is moving due west, which implies that u must be given the negative sign. The results are therefore as follows:

$$s_0 = \left(6 - 10 \times \frac{4}{5}\right) \Big/ \frac{3}{5} = -3\tfrac{1}{3}$$
$$t_0 = \left(10 - 6 \times \frac{4}{5}\right) \Big/ \frac{3}{5} = 8\tfrac{2}{3}$$

It may have been noticed that in several instances u has been selected with such a value that $1 - u^2$ is an exact square. This has been done to simplify the computations, but in many of the examples which follow the above expression will not be an exact square. In most cases accuracy to the first two decimals will suffice for our purpose.

Two more examples are given, and the reader should work these out

for himself by using the above formulae. These are very important, as certain conclusions which are based on them must be understood before proceeding further.

Another observer says that A's world is moving away from him due east ~~west~~ with a velocity 0·3. What are his records, assuming that A says the intervals between two events are the same as before (6, 10)? (The space- and time-intervals will be denoted in this way for convenience.)

Answer. He records the intervals as (9·43, 12·37).

Another observer says that A's world is moving from him due east with a velocity of 0·25. What are his records?

Answer. His records are (8·78, 11·88).

The Separation of Events

The four results are obtained on the assumption that the two events had intervals of (6, 10). Naturally different results would be obtained if the intervals were altered, but we shall adhere to the same figures for the present. The results are shown below:

Value of u	0·4	−0·4	0·3	0·25
Distance-interval	23·33	−3·33	9·43	8·78
Time-interval	24·67	8·67	12·37	11·88
$t^2 - s^2$	64	64	64	64

These four examples will be sufficient to show that there is an interesting relation between the distance- and time-intervals. If we deduct the square of the space-interval from the square of the time-interval we obtain the figures shown in the last row, decimals being ignored. It will be seen that the figures obtained are the same as those found by deducting the square of the space-interval from the square of the time-interval in A's world, that is $10^2 - 6^2 = 64$. Although the time- and space-intervals vary in the different worlds, nevertheless all the observers agree that $t^2 - s^2$ is constant and is 64. If we had started with the original intervals as 12 and 4 say, we should have obtained $12^2 - 4^2 = 128$ as the constant, whatever values of u were used.

The expression $\sqrt{(t^2 - s^2)}$, which will be represented by S, is called the *separation* of the two events, and is a fusion of space and time. It is quite independent of the world in which the records are made and represents an intrinsic property connecting the two events, irrespective of the conditions under which they were observed. This may seem a little confusing, but a few simple illustrations will clarify the subject.

Light travels *in vacuo* with a velocity of 186,271 miles a second, and hence requires 500 seconds to travel from the sun to the earth when the sun is at his mean distance from the earth (about 93,005,000 miles). Suppose event 1 occurs on the sun and event 2 on the earth, event 1

ing a solar eruption and event 2 being the appearance of a solar ominence. Let the interval be 500 and 400, 500 being the space-:erval and 400 the time-interval. This implies that the distance of the n is 500 light-seconds (the space travelled by light in 500 seconds) and at the time between the events is 400 seconds. In this case $t^2 - s^2 =$ 90,000, and as this is negative, its square root is imaginary. This es not mean that the events are imaginary but it has an interpretation iich is important.

The message sent off from the sun requires 500 seconds to reach e earth and hence it could not arrive at the earth where event 2 took ace before the occurrence of event 2, because the time-interval of ent 2 was only 400. Hence no time order exists in this case. It may pointed out that when an observer places on record the time of an ent he gives the corrected time after allowing for the time that the ;ht requires to reach him, and he can do this when he knows the stance where the event takes place. Suppose, for example, that the ginning of an eclipse of the sun is observed at 11^h. To find the time which it really commenced the astronomer must make allowance for e time that light requires to reach the earth, and so he would deduct minutes 20 seconds from 11^h to obtain the time at which the eclipse tually commenced.

Suppose in the next case that the time-interval of event 2 is 600, en S^2 is positive and a time order exists. Soon after event 1 has appened on the sun we can imagine a wireless message sent off to the rth reporting the event (the wireless message will travel with the speed light) or simply a light signal announcing an eruption, and this will ach the earth in 500 seconds. Since event 2 took place with time-terval 600 seconds it is easily seen that the message about event 1 ill reach the earth before the occurrence of event 2.

Suppose S is zero, what interpretation shall we give in this case? bviously in such circumstances $t = s$, or, in other words, the time-interval etween the events is the same as the time required by a light-signal to avel from the sun to the earth. This merely shows that the signal was nt off from the sun as soon as event 1 took place and was observed n the earth as soon as it arrived there. It is clear that it could not ave been seen a second sooner.

It is important to remember that all observers, if we could imagine iem on different planets and moving with various velocities which, for he sake of illustrating the point, can be taken as very great, would make ifferent records of t and s. Their values for these could be found from he equations previously given, provided u were known in each case. t is equally important to notice that each observer is entitled to his iew and that there is nothing to show why any preference should be iven to the opinion of one more than another. When we deal with he ordinary velocities with which we are accustomed on the earth, the

views of various observers are nearly the same—so close indeed that it is generally impossible to detect any difference. Nevertheless such differences exist, though we have been unaware of them until comparatively recent times. We shall now use an illustration which is not purely imaginary, in which a fairly high velocity is involved.

Fig. II (6) (a) shows the earth E, Jupiter J, and a distant spiral nebula N which has a star in it attended by a planetary system. An observer on one of these planets says that the solar system is receding from him at a speed of 1860 miles a second (not an improbable velocity if we imagine that the spiral nebula is about 20 million light-years distant). An observer on E notices two special events: (1) an eclipse of one of Jupiter's satellites; (2) a light or wireless signal from Jupiter which he receives 3000 seconds after event 1. How does the observer on the planet somewhere in the spiral nebula record the interval between the events? The distance of the earth from Jupiter can be taken as 2600 light-seconds.

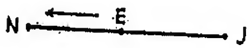

FIG. II (6) (a)
How an observer on a planet in a distant nebula which is receding from the earth records an event on Jupiter.

FIG. II (6) (b)
The same as Fig. 6 (a) except that Jupiter is now between the earth and the nebula.

Using the formulae deduced on p. 182 and noticing that the observer within the nebula corresponds to O and that u is 0·01 because the velocity of E relative to N is 1/100 the velocity of light, we find as follows:

$$s_0 = (2600 + 0{\cdot}01 \times 3000) \times 1{\cdot}00005 = 2630 \text{ to four significant figures}$$
$$t_0 = (3000 + 0{\cdot}01 \times 2600) \times 1{\cdot}00005 = 3026$$

The approximate value of $1/\sqrt{(1-u^2)} = 1 + \frac{1}{2}u^2$ has been used and is sufficiently accurate for the present purpose.

The separation in this case is $\sqrt{(3026^2 - 2630^2)} = 149{\cdot}7$ to four significant figures, and this is practically the same as $\sqrt{(3000^2 - 2600^2)}$, as we should expect, because it has been shown that the separation is the same for each observer. The very slight discrepancy between the two values of the separation is due to the fact that only four figures were used in the computation of s_0 and t_0.

What would O's opinion be if the relative positions were as shown in II (6) (b)?

In these circumstances it is necessary to make $u = -0{\cdot}01$. On substituting this value it is easily found that $s_0 = 2570$, $t_0 = 2974$. The separation is $\sqrt{(2974^2 - 2570^2)} = 149{\cdot}7$ as previously obtained.

Suppose that the observer on the earth receives the signal 2610 seconds after event 1, and that the relative positions of *N*, *E*, and *J* are as shown in (b). How does the observer in the spiral nebula record the interval between the events?

$$s_0 = (2600 - 0{\cdot}01 \times 2610) \times 1{\cdot}00005 = 2574$$
$$t_0 = (2610 - 0{\cdot}01 \times 2600) \times 1{\cdot}00005 = 2584$$

These results should be noticed very carefully as they involve an apparent contradiction.

Considering the last example for the present, what does it show us regarding time-sequence? We have seen that an observer on the earth receives the signal 2610 seconds after event 1, which implies that he received it 10 seconds after the eclipse. The observer on the planet within the nebula judges that the time was only 2584 seconds, or in other words, according to him event 1 followed event 2, and this involves an *apparent* contradiction. If the reader will substitute the value $-0{\cdot}00385$ for u he will find that $t_0 = 2600$, so that an observer on a planet in a nebula which had the velocity $0{\cdot}00385$ would judge the events 1 and 2 to be simultaneous. The special theory of relativity shows us that there is really no such thing as before or after or simultaneity when bodies are moving relative to other bodies. It all depends upon the point of view of each observer and no one can claim the right to be more correct than another. While this may seem a startling view, it must be remembered that it is only startling because we have been accustomed to judge from the standpoint of a universal cosmic time. For each body there is a time order of events which has been called its "Proper Time", and the proper time varies according to circumstances. So far as our own experience is concerned this is always governed by the proper time for our own body. It may be admitted that the proper times of human beings are very nearly the same, but this is only because our speeds relative to one another are very small in comparison with the speed of light, and so, for all practical purposes, the proper times for all of us can be taken to be the same and can be identified with terrestrial time.

It may be objected that all this may be useful for the metaphysician but that it has no bearing on our ordinary life. Even if it is admitted that people on planets which have high speeds with reference to the solar system have their own proper times, there is nothing on our own planet comparable to this. In answer to this it may be pointed out that when we come to deal with the electrons later in this work, it will be shown that the relativity theory has a most important bearing. In addition, it will be shown that in the solar system itself the general theory of relativity has some very relevant applications.

Before proceeding to the next chapter the reader is advised to make

himself familiar with the application of the formulae given on pp. 181–3 by solving the problems given below. A positive value for u can be assumed in 2 and 3.

Problems

1. A gives the interval between two events in the form (2, 3), and O says that A's universe has a velocity of 0·3. How does O record the interval? What is the separation?

Answer. (3·04, 3·77); 2·24.

2. A records the interval between two events as (7, 10), and O says that the time-interval is 14 seconds. Find (1) the velocity that O attributes to A; (2) O's record of the space-interval; (3) the separation.

Answer. (1) 0·4; (2) 12; (3) 7·14.

3. A records the interval between two events as (5, 7), and O says that the space-interval is 9·81. What velocity does O attribute to A and what is O's record of the time-interval?

Answer. 0·5; 11 nearly.

4. The following events are noted on the same day: (a) an earthquake at Formosa at 1^h; (b) an eclipse of a satellite of Jupiter at $1^h\ 30^m$; (c) occultation of Aldebaran by the moon at $1^h\ 55^m.0$. What do you know about the time-order of these events?

Answer. (c) occurred after (a) and (b); no time order exists for (a) and (b).

In 1 to 3 u is considered positive. Notice that a quadratic equation is involved in 2 and 3 and the positive values of the solutions are given in the answers.

In 4 use the corrected times, allowing for the time light requires to travel from Jupiter to the earth and also from the moon to the earth.

Chapter IV

THE WORLD OF THE FLATLANDER

Up to the present we have considered events which take place at points on a straight line along which the worlds are separated, and it is now necessary to extend this to deal with events which occur anywhere in space. Most readers have probably a knowledge of three-dimensional geometry, but for the sake of those who are not conversant with it the following elementary explanation will be sufficient for all that is contained in this chapter.

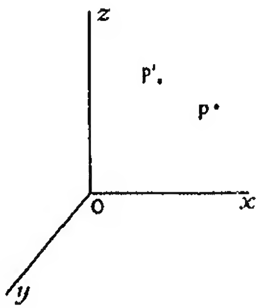

Fig. II (7)
How to find the distance between two points the co-ordinates of which, referred to three planes, are given.

A point P in a room, say an electric bulb, can be defined by referring it to its distances from two walls and the floor, as shown in Fig. II (7). These are the planes of reference, and if its distances from these planes are 10, 8, and 7 feet, then its distance from O is $\sqrt{(10^2 + 8^2 + 7^2)} = 14{\cdot}6$ feet. This is merely an extension of the theory of Pythagoras which says that the square on the hypothenuse of a right-angled triangle is equal to the sum of the squares on the other two sides. In addition, if another point P' is taken whose distances from the planes are 6, 3, and 9, respectively, the distance between P and P' is

$$\sqrt{\left((10 - 6)^2 + (8 - 3)^2 + (9 - 7)^2\right)} = \sqrt{45} = 6{\cdot}7 \text{ feet.}$$

Now suppose that an event A is given by $x = 10, y = 8, z = 9, t = 20$. This means that it is located by its distances from three planes at right

angles to one another, the distance x being measured from O towards the right, that of y being measured perpendicular to the plane of the paper, and the distance z being measured vertically. The axes Ox and Oz are in the plane of the paper. The time-interval of 20 cannot be represented as a fourth dimension but it will be shown later how to deal with it in a simple way.

If an event B is given by $x = 6, y = 3, z = 7, t = 8$, the space-interval between A and B is 6·7, as shown above. The time interval is 12 $(20 - 8)$ and hence the separation is $\sqrt{(144 - 45)} = 10$ approximately. Problems of this kind are treated on the same principles as those where only one co-ordinate was considered (p. 184) and do not present any special difficulties.

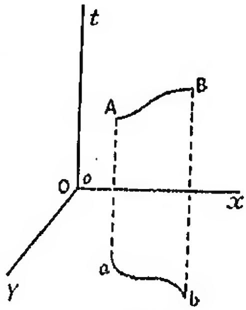

FIG. II (8)
A Flatlander in a three-dimensional world, time being a dimension.

If we imagine that the points P and P' or, if we wish, the events A and B, are in the plane xOy, which is the plane of the floor, the same method is adopted. Thus, suppose that the co-ordinates of x and y are the same as before and also the time-intervals; the distance PP' is now $\sqrt{((10 - 6)^2 + (8 - 3)^2)} = \sqrt{41}$, and hence the separation is $\sqrt{(144 - 41)}$, which is slightly greater than 10.

We shall now deal with an imaginary being who is a Flatlander living in a world of two dimensions as shown by xOy, Fig. II (8). From O draw Ot perpendicular to the plane xOy and let Ot represent the time-axis. It will be seen that we have dispensed with the z-axis because it is a two-dimensional world so far as space is concerned, and we can easily visualize the third axis as representing the time-axis. Let us follow the movements of Flatlander, whom we shall describe by F in the future.

F starts his life at a, and an observer O at o makes a record of his life history. He does this by finding out how far F is from Oy and also from Ox at any instant, these distances being denoted by x and y respec-

y, and, in addition, he makes records of the times, so that he can e use of the t-axis also. O can therefore represent each event in the of F by a point in space, not in the plane xOy but as shown in the e which includes t. Thus the point A corresponds to event a, the t B to event b, and so on, so that the history of F is represented by curve which we can call F's "world-line". There may be thousands illions of Fs, each one of whom has his own world-line, and, as shown ig. 8, these make up the space-time of the Flatlanders' universe.

Suppose that two Fs collide, or better, suppose there are two events, the marriage of one F and the death of another. O will record these he intersection of two world-lines, and if he wants to compile a logue of simultaneous events he must select points which are at the e height above the plane xOy, or simply points which have the same es for the t-co-ordinate. In most cases all the Fs will agree closely . O's conclusions, but if one F was capable of moving rapidly he would e different space and time measurements from O.

Let us now take a numerical example from the Flatlanders' universe, we shall concentrate our attention on one which can be taken as cal of all the others. The units are the same as those previously oted.

Event	A	B	C	D	E
Time t	5	10	20	40	80
x-co-ordinate ..	1	2	4	8	16
y-co-ordinate ..	3	6	12	24	48

The time-interval between the events E and A is 75 (80 − 5).

The space-interval of E from A is $\sqrt{((16-1)^2+(48-3)^2)} = \sqrt{2250}$ 47·43.

The separation of E from A is $\sqrt{(75^2 - 47{\cdot}43^2)} = 58{\cdot}1$.

The space-interval from E to A is the sum of the space-intervals of o B, B to C, C to D, and D to E, and the same applies to the time-rval. Thus, if we consider the x-co-ordinate, the sum of the space-rvals from A to B and so on is $1+2+4+8=15$, and in the same r it is seen that this applies to the y- and t-co-ordinates also. Does the e apply to the separation? Testing this, we obtain the following ılts:

Intervals	A to B	B to C	C to D	D to E
-ordinate	5	10	20	40
-ordinate	1	2	4	8
-ordinate	3	6	12	24
ıares of t-intervals ..	25	100	400	1600
,, x-intervals ..	1	4	16	64
,, y-intervals ..	9	36	144	576

Intervals	A to B	B to C	C to D	D to E	
Sum of squares of x- and y-intervals	10	40	160	640	
Squares of t-intervals minus the last row ..	15	60	240	960	
Separations	3·87	7·75	15·49	30·98	
Sum of separations ..	..	..	..	..	58·09

We have already shown that the separation between A and E is 58·1, a discrepancy of only 0·01 occurring owing to an accumulation of small errors. This equality will hold under certain conditions which will be now considered.

It will be seen that if each of the x-co-ordinates is divided by the corresponding t-co-ordinate the result is 1/5. Also, if each of the y-co-ordinates is divided by the corresponding t-co-ordinate the result is 3/5. Expressed in a different way we can say that each of the rows is a series in geometrical progression with the same common ratio—in the present case 2. It makes no difference what the common ratio is so long as it is the same for each row. In such circumstances the separation between the first and the last event will always be the sum of the separations between the first and the second, the second and the third, and so on to the last.

Does this rule hold if the common ratio is not the same for each co-ordinate? To answer this question a test will be made from another specific example, and two of the rows will be assigned the same common ratio which, however, will differ from that of the third row.

Event	A	B	C
t-co-ordinate	5	10	20
x-co-ordinate	1	3	9
y-co-ordinate	3	6	12

Intervals	A to B	B to C	A to C
t-co-ordinate	5	10	15
x-co-ordinate	2	6	8
y-co-ordinate	3	6	9
Sum of squares of x- and y-intervals	13	72	145
Difference between squares of t-intervals and last row	12	28	80
Separation	3·46	5·29	8·94

The sum of the first two separations is 8·75, which is less than 8·94, the separation between events A and C, and however many cases are taken it will be found that the separation of the last event from the first is always greater than the sum of the separations of the first from

the second, the second from the third, and so on to the last. This holds only under the conditions that the common ratio referred to shall not be the same for each co-ordinate.

The interpretation of the above results is not difficult. If the reader will plot the curve in the first case between any two of the co-ordinates, say x and y, then x and t, then y and t, he will find that in each case it is a straight line. If he does the same in the second example he can obtain a straight line if he uses the t- and y-co-ordinates but in no other case. In the first example O says that the Flatlander is moving with uniform speed in a straight line, which implies that he is moving freely, being uninfluenced by any force. In the second example the Flatlander's world-line is curved and he is not moving freely. The meaning to be assigned to the terms "freely" will be discussed later.

In Fig. II (9) let the world-line of F consist of two straight portions AB and BC. From what has just been said we know that the separation of C from A is greater than the sum of the separations of B from A and

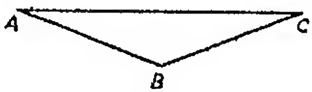

FIG. II (9)
Deals with "separations". See text for explanation.

of C from B. This seems contrary to Euclidean geometry, which says that AB plus BC is greater than AC, but we are not now dealing with Euclidean geometry. Any number of paths could join A and C in space and time, but AC is unique in one respect—all observers agree that it yields a separation greater than any of the others. The separation of the various paths would differ from one another but any one would be less than AC.

Although we have been dealing with a race of Flatlanders the same argument applies in three-dimensional space. The name *geodesic* is applied to the world-line possessing the unique property referred to—that which yields the maximum separation. We have seen that the separation between two events in the life of a body is equal to its "proper time" which is the time interval measured by a clock which the body carries about with it. It appears, therefore, that if a body is left to itself it will follow the path which makes the proper time between events as great as possible, according to its own clock. Let us see how all this compares with Newton's laws.

Newton presupposed absolute space, time, and motion, and it must be admitted that those who have been brought up on Newtonian mechanics find it difficult to free themselves from their bondage. He said that a body left to itself moved in a straight line, but now we must ask

ourselves what we mean by a "straight line". A line which is straight in one person's space may be curved in the space of another person. Then again, in his second law Newton measured force by the acceleration or rate of change of motion of a body, but in whose system is the rate of change to be measured? What did Newton mean by force which he could not observe but which he postulated? In the Newtonian sense we cannot observe force, which is a mere hypothesis, though we can observe change of motion. The rate of change of motion certainly implies a cause, and probably no serious harm is done by formulating a hypothesis to describe it, so long as this hypothesis is recognized as purely provisional. A relativist can state Newton's first law in a different form, which does not involve so many difficulties and ambiguities, as follows:

"If a body is moving freely and if A and B are two events in its history, then the space-time path which the body follows between A and B is such that the separation of B from A, measured along that path, is a maximum."

It will be necessary to return to this point later in the work when we come to deal with general relativity. Before proceeding to deal with the problems of mass and momentum arising out of the previous investigation, a few examples will be worked to make the reader familiar with the formulae employed. Problems follow which the reader can then work out for himself.

Example 1

An event A is given by $x = 2, y = 3, z = 4, t = 20$, and an event B by $x = 1, y = 5, z = 7, t = 25$. What is the separation of B from A? (In future the co-ordinates will be written in the form $(2, 3, 4; 20)$, etc.)

The differences between the co-ordinates of x, y, z, and t respectively, irrespective of the signs, which will not affect the results, are 1, 2, 3, and 5. The separation is, therefore $\sqrt{(5^2 - (1^2 + 2^2 + 3^2))} = \sqrt{11} = 3{\cdot}32$.

Example 2

If the world-line of a particle is the straight line AB in the above, what are the space co-ordinates of an event happening to the particle when $t = 30$?

A change of 5 $(25 - 20)$ in t implies a change $(-1, 2, 3)$ in the other co-ordinates. A change of 10 $(30 - 20)$ is required by the problem, and hence the change in (x, y, z) is twice that given above, or $(-2, 4, 6)$. Adding these to the first co-ordinates the result is $(0, 7, 10)$.

Example 3

A, B, C, events in the life of a particle, are given by (0, 0, 0; 0), (3, 6, 14; 20), (7, 9, 16; 25). What is the separation of C from A? Is the particle moving freely?

The separation is $\sqrt{(25^2 - (7^2 + 9^2 + 16^2))} = \sqrt{239} = 15 \cdot 46$.

From what has been previously said about the criterion for a particle moving freely it is obvious that in this case it is not moving freely. This can be checked by noticing that the separation of C from A is not the same as the sum of the separations of B from A and of C from B.

Problems

1. Find the separation between events given by (3, 6, 10; 12), and (2, 3, 4; 20).

Answer. 4·24.

2. Verify that a particle is moving freely from the following co-ordinates of three events: (3, 4, 5; 10), (9, 12, 15; 30), (27, 36, 45; 90).

3. An observer says that the events in 1 occur at the same place. What time-interval does he attribute to the two events?

Answer. 4·24 seconds. (Notice that the separation remains the same.)

CHAPTER V

VELOCITY AND MASS IN DIFFERENT WORLDS

Composition of Velocities in a Moving World

IT will be necessary at this stage to refer to a previous example given on pp. 182–3, which will be stated in a slightly different form as follows:

An observer O says that A's world is moving away from him with a velocity $4/5$. In A's world there are two special events: (1) a ball passes the position A, with a velocity 0·6 moving in the same direction as A's world; (2) the ball is 6 units from A 10 seconds later. What velocity does O attribute to the ball?

It will be seen that we have derived the velocity of the ball in A's world by dividing the space traversed, 6 units, by the time, 10 seconds.

It was shown that O attributed a space-interval $23\frac{1}{3}$ and a time interval $24\frac{2}{3}$ to the ball, and hence he asserts that its velocity is $23\frac{1}{3}/24\frac{2}{3} = 70/74$.

Now let us use the ordinary method for finding the velocity of the ball.

A's velocity from O is 0·8 and the velocity of the ball relative to A is 0·6, so that the velocity of the ball relative to O is $0{\cdot}8 + 0{\cdot}6 = 1{\cdot}4$. This differs from the velocity found above, which is only $70/74$, and it is obvious that the old method used for the composition of velocities is erroneous. Without dealing with the method of proof it may be said that the formula which must be used when compounding velocities in the same line is as follows, where u is the velocity of the world in which the event occurs, v is the velocity of the body in this world, according to an observer who is moving with it, and w is the resultant velocity,

$$w = (u + v)/(1 + uv).$$

Substituting 0·8 and 0·6 for u and v in this formula, we find

$$w = 1{\cdot}4/1{\cdot}48 = 140/148 = 70/74.$$

When the velocities are in opposite directions the negative sign must be used with one of them—preferably with the smaller velocity.

If the velocity above is 1, that is, if the body in A's world is moving with the velocity of light, $w = (u + 1)/(u + 1) = 1$, that is, the resultant velocity is the velocity of light. This is just what we should expect, because, as we have seen earlier, all observers who measure the velocity of light, whatever their own velocities may be, obtain the same result.

The Newtonian method of composition of velocities is not strictly

accurate, but when we are dealing with the velocities to which we are accustomed in our world it is difficult to detect any discrepancy. Generally speaking, our velocities are very small in comparison with that of light, and hence w will differ very little from $u + v$. An example will make this clear.

Suppose an observer O is at rest relative to the sun and hence says that the earth is moving with a velocity of 18½ miles a second. Imagine that a train is travelling at a speed of 60 miles an hour in a direction opposite to that of the earth's orbital motion. How will O judge the speed of the train?

Expressing all speeds in terms of that of light, $u = 0{\cdot}0001$, $v = -0{\cdot}00000009$, and hence $w = (0{\cdot}0001 - 0{\cdot}00000009)/(1 - 0{\cdot}000000000009)$ the negative sign being used as the velocities are in opposite directions. The numerator is obtained by the usual Newtonian method, and the discrepancy between this method and the more accurate method appears in the denominator. As will be seen, this discrepancy is less than 1 in ten thousand million and, even with the velocity of 18½ miles a second for the earth and 60 miles an hour for the train, would be only of the order of the one-thousandth of an inch per second.

There is an important verification of the formula for the composition of velocities. Up to the present we have considered velocities *in vacuo*, but when light is propagated through any medium its velocity differs from that *in vacuo*. Its velocity in air is nearly the same as *in vacuo* because the refractive index of air is nearly 1, and the velocity varies inversely as the refractive index of the medium. In the case of water with refractive index 4/3 the velocity of light is 3/4, that *in vacuo* being the unit.

Suppose light is transmitted through a stream of water which is moving through a tube with velocity u in the same direction as the ray of light. Can we deduce the velocity of the ray relative to the tube? Using the formula given for the composition of velocities,

$$w = \left(\frac{3}{4} + u\right) \Big/ \left(1 + \frac{3}{4}u\right)$$

Since u is very small when it is expressed in terms of the velocity of light, the value of $1/(1 + \frac{3}{4}u)$ in the above expression is practically $1 - \frac{3}{4}u$, and hence, multiplying this by $\frac{3}{4} + u$, we find

$$w = \frac{3}{4} + u - \frac{9}{16}u - \frac{3}{4}u^2$$

The last term involving u^2 is so small that it can be ignored, and the final value for w is $\frac{3}{4} + u\left(1 - \frac{9}{16}\right)$.

If μ is used instead of 4/3 to denote the refractive index of the medium, the value of w can be written in the form

$$w = \frac{1}{\mu} + u\left(1 - \frac{1}{\mu^2}\right),$$ which can be used for any medium.

Experiments by Fizeau in 1851 and by Hoek in 1868 showed that the rate of advance of the light-ray relative to the tube was in accordance with the above formula, very close approximations to the theoretical results being obtained.

The Mass of a Body in Motion

It is a little difficult to deal fully with the problem of mass in an elementary treatise, and the reader must be prepared to accept certain conclusions without adequate proofs. If we agree to all that has been said up to the present about the special theory of relativity, we may conjecture that the mass of a body is not independent of its velocity. It will suffice to say that, just as length, time and velocity are different in different worlds in motion relative to one another, so masses are different also.

Suppose the mass of a body at rest in A's world is m and then that it moves in A's world with a velocity u, A will measure its mass as

$$m/\sqrt{(1 - u^2)}$$

Since u is usually very small the above expression is approximately $m\ (1 + \frac{1}{2}u^2)$, and those who have an elementary knowledge of mechanics know that the second term, $\frac{1}{2}mu^2$, represents the kinetic energy of the body. Thus a body with mass 100 gm. moving with a velocity of 200 cm. per sec. has a kinetic energy or capability of doing work represented by

$$\tfrac{1}{2} \times 100 \times 200^2 = 2 \text{ million ergs.}$$

It is now accepted that mass is nothing other than a form or appearance of energy, and annihilation of matter implies a certain amount of energy released in the form of radiation. The amount of energy thus released by a mass of m gm. is mc^2 ergs, c being the velocity of light in cm/sec. Hence each gramme of matter is equivalent to $(3 \times 10^{10})^2$ ergs, irrespective of the time required for the annihilation of the matter. If a body of mass 1 gm. moves with a velocity of 200 cm. per sec., its total energy is, therefore $(9 \times 10^{20} + 20{,}000)$ ergs.

We can represent the total energy of a body, potential and kinetic, by the expression $mc^2\ (1 + \frac{1}{2}u^2)$, so that the mass of a body moving with velocity u is $m\ (1 + \frac{1}{2}u^2)$ which is very nearly the same as $m/\sqrt{(1 - u^2)}$ when u is small. The fact that the mass of a body increases with its velocity merely tells us that an increase in its kinetic energy reveals itself by an increase in the apparent mass. If u could become 1, that is, if the body could move with the velocity of light, the mass would be infinite, as the denominator in the above expression would be zero. No body can attain the velocity of light and in fact the expression for the mass of a body sets an upper limit to the velocity of any body. Since m

can never be infinite it follows that u can never attain the value 1. The nearest approach to the velocity of light takes place with electrons, and it has been known for a long time, before Einstein propounded his relativity theory, that electrons moving with high speeds increased their apparent mass by the amount suggested by the above expression.

The energy of a body at rest is, as we have seen, mc^2, and this has been called the "energy of constitution" of the body. If we regard the energy of the sun as due to the "annihilation" of matter, we must conclude that the sun, like other stars, is losing mass. A simple calculation shows that on this view the present output of solar energy requires the annihilation of about 4 million tons per second. This may seem very large, but considering that the mass of the sun is about 2×10^{27} tons, it is relatively very small.

The view that the output of energy of the sun and other stars is due to the annihilation of matter has been confirmed in recent years. The transformation of hydrogen into helium, induced by the high temperature in the interior of the sun and many other stars, and aided by the catalytic action of carbon and nitrogen, is believed to supply the necessary energy, a certain amount of mass disappearing in the process. A discussion of this, however, is outside our scope.

Example 1

A body is moving in A's world with a velocity 0·3 in the direction A to B. O says that A's world is moving in the direction A to B with a velocity 0·4. What is the velocity of the body according to O?

Substituting 0·3 for v and 0·4 for u, the formula for w gives $0{\cdot}7/1{\cdot}12 = 0{\cdot}58$.

Example 2

If O says that A's world is moving in the direction B to A, what velocity does he attribute to the body?

In this case v is $-\ 0{\cdot}3$ and u is 0·4, so that w is $0{\cdot}1/0{\cdot}88 = 0{\cdot}114$. The direction of motion, according to O, will correspond with that of u, which is from B to A.

Example 3

O says that the velocity of a particle in A's world is 5/17 in the direction AB and also that A's world is moving in the same direction with a velocity 0·1. What does A say the velocity of the particle is?

$$w = 5/17,\ u = 0{\cdot}1, \text{ hence } 5/17 = (0{\cdot}1 + v)/1 + 0{\cdot}1\, v), \text{ from which}$$
$$5/17 - 5v/170 = 0{\cdot}1 - v, \text{ or } 165v/170 = 33/170,$$
$$\text{hence } v = 0{\cdot}2.$$

Since the value of v is positive, A says that the particle is moving in the direction AB.

Example 4

If an electron is moving with a velocity 0·4 verify from the exact expression for the mass of a particle in motion that its apparent mass increases by more than 9 per cent. What increase is given by the approximate formula?

The mass is $1/\sqrt{(1 - u^2)} = 1/\sqrt{0{\cdot}84} = 1{\cdot}091$ if the mass at rest is 1, though, strictly speaking, there is no such thing as an electron at rest. When we use the term "at rest" it implies a small velocity.

The approximate expression gives $m + \frac{1}{2} \times 0{\cdot}16 = 1{\cdot}08$. In the case of these high speeds it is better to use the exact formula.

Problems

1. A body at rest has a mass 2. It then moves in A's world with a velocity 0·3. What is A's measure of its mass?

Answer. 2·097.

2. O says that A's world is moving in the same direction as the body in 1, with a velocity 0·1. What is O's measure of the mass?

Answer. 2·18.

3. A body in A's world is moving with a velocity of 0·5 and O says that its velocity is 0·421 in the same direction. How does O judge the velocity of A's world?

Answer. O says that it is moving in the opposite direction with a velocity 0·1.

4. A's world is moving with a velocity 0·5 and a body in his world is moving with a velocity 0·5 in the same direction with reference to O. Why does O not think that the velocity of the body with reference to himself is the same as that of light?

Answer. The denominator of the expression for w exceeds 1.

5. What is the mass of a body whose mass is 1 at rest, as judged by A and O respectively in 4?

Answer. 1·15 and 1·67.

6. The mass of an electron at rest is about 8×10^{-28} gm. With what velocity must an electron move, in kilometres a second, so that its apparent mass may be (a) 12×10^{-28} gm., (b) 24×10^{-28} gm.?

Answer. (a) 223,000, (b) 283,000.

7. What would be the loss of a watch per day and the shortening of a foot rule in 6 (a)?

Answer. 8 hours; 4 inches.

Chapter VI

SUMMARY OF THE RESULTS OF SPECIAL RELATIVITY

A summary of the position may assist the reader at this stage if he has understood the significance of the new conception of the physical world and also the elementary formulae which embody that conception. Some may think that a summary in the first instance would have been more helpful, but this is a mistake. Many popular accounts of the theory of relativity which are free from any form of mathematics have not always been successful in enlightening the reader. When the new ideas are expressed in non-mathematical language they are still difficult—probably more difficult than they would be if mathematics were introduced. If the subject has been followed carefully up to the present it will be obvious that, to a large extent, the theory of relativity depends on throwing overboard a number of conceptions which are wrong, though they work fairly well, and hence have come to be regarded as necessities of thought.

It must be borne in mind that the Universe cannot be completely comprehended by our finite minds, though it can be interpreted. This interpretation depends on ourselves and our faculties. Science is conditioned by the human mind and must therefore be relative to it. We must not, however, fall into the error of asserting that everything is relative; if this were true there would be nothing in the Universe to which it could be relative. It is true, on the other hand, that everything in the physical world is relative to the observer, and for this very reason the theory of relativity seeks to exclude what is relative and to arrive at statements of physical laws that shall be independent of the observer. If it failed to do so it could not claim to be science.

The Michelson-Morley experiment shows that the velocity of light *in vacuo*, as determined by every individual, is an absolute constant—a statement which seems extraordinary from the point of view of tradition and "common sense". If a number of people walk along a road at different speeds and a number of motor-cars dash past them, people and cars going in different directions, in a few seconds they will be at different distances from a point on the road if all started there at the same instant. This is mere common sense, but if we apply our common sense to the next step in the argument it will seem to contradict the relativity theory. Suppose a flash of light is sent out at the instant

when they are all at the same point, the light-waves will be at 186,271 miles from each pedestrian and car a second later, by each one's clock. This seems to be impossible by our conventional way of thinking, because in the second some of the cars might be 50 feet from the point on one side and some the same or a greater distance from it on the other side, and the pedestrians, too, would be at various distances from it. If the reader has followed the results of the Michelson-Morley experiment and also its application to the illustration of the men in the boats, he will see that this is what relativity leads us to—each observer will find that the velocity of light is precisely the same.

We have been accustomed to regard matter, space, and time as the three independent foundation-stones of the Universe, and indeed Science has been obliged to adopt them as the data in terms of which discoveries can be expressed. But now men of science have good reasons to enquire whether they are the absolute and fundamental things that they were once considered to be. Suppose that they are not absolute but mean different things to different people? If A calls a certain interval a minute and B calls it half a minute, or if A says that the length of an object is a foot and B says that it is half a foot, and if there is no criterion for testing the validity of each one's statement, we need not be surprised if apparently contradictory results are obtained. Nevertheless, if we regard the Universe in the right way we shall see that failure to detect absolute motion is nothing more than an observable natural occurrence, and once we have convinced ourselves that absolute motion is meaningless, we shall find no difficulty in calculating the necessary changes that must be introduced in certain terrestrial standards.

If absolute motion is meaningless, why should we have expected to be able to measure it and how does this knowledge affect our standards? The answer to the first question is that we have entertained false conceptions of the Universe in the past, and when we have discarded these, Nature is simplified. As Sir Arthur Eddington says, "The relativity standpoint is then a discarding of certain hypotheses, which are uncalled for by any known facts, and stand in the way of an understanding of the simplicity of nature."* We have already answered the second question when it was shown how our conceptions of length, time, and mass were modified. Let us return to the definition of length given on p. 177.

There seems something very arbitrary in defining the length of a body as $l\sqrt{(1 - u^2)}$, where u is the velocity of the body in the direction in which the length is measured, with reference to the standard of rest adopted. The view that this definition is arbitrary arises from our earlier outlook when we thought in terms of Newtonian mechanics. This outlook was responsible for the conception of length as absolute and it is not easy to free ourselves from the old obsession. Here is an example

* *Space, Time and Gravitation*, p. 29.

of a false view in another sphere which we have discarded without any difficulty.

Everyone knows, or thinks he knows, the meaning of the term "weight". When two bodies have the same weight this fact is indicated by a good balance of the usual type or by a spring balance, and we shall confine our attention to the latter for the present. Suppose we are given a pound weight of some commodity and we check it on a spring balance say at a place in the latitude of Greenwich. It might not occur to everyone that the weight is not an intrinsic quality of the body, but if we experimented on different places on the earth's surface we would find that there was nothing absolute about the weight of the body. If we could test it at either Pole by means of the spring balance we would find that it weighed 1·003 lb. and if we went to the equator it would weigh just under 0·998 lb. If we could take it to the moon it would weigh about 1/6 lb., while on Jupiter it would weigh more than $2\frac{1}{2}$ lb. In fact, it would show a different weight on every planet or satellite, and, as has been shown, even on the earth itself there is nothing absolute about the weight of the body.

If we were anxious to define the weight of a body with greater accuracy we would discard some of the old conventions and ideas about the permanency of weight and would proceed as follows:

The weight of a body on the earth will be defined by the expression

$$m_0 \,(1 - 0{\cdot}00265 \cos 2\,\phi)$$

where m_0 is its weight at latitude 45° and ϕ is the latitude of the place. Although this expression neglects small terms and is not, therefore, exact, it is a very close approximation and will suffice for the purpose of the illustration.

If the weight of a body cannot be regarded as absolute why should there be any reason for treating the length of a body as an intrinsic property of the body? It has been shown why lengths l in one world are measured as lengths $l\sqrt{(1 - u^2)}$ in another world, u being the relative velocity of one world with reference to the other. As Professor H. Dingle points out; "The special theory of relativity is *completely* contained in the purely physical statement that the fundamental measurement of physics is $l\sqrt{(1 - v^2/c^2)}$, all other measurements which in classical physics have been defined in terms of l being thereby subject to modification only by the substitution of this more complete expression, their definitions remaining otherwise the same."* (It should be noticed that v is the velocity of the body and c that of light, so that v/c corresponds to u, which has been used in the present work.)

The modification in time corresponding to that in length is easily derived. In the description of the conversation between A, B, and C,

* *The Special Theory of Relativity*, pp. 29–30.

given on p. 175, it was shown that B's clock must lose to compensate for the shortened course, and this loss was proportional to the shortening of the course. Velocity is simply length divided by time, and if velocity is to remain unchanged, while length becomes $l\sqrt{(1 - u^2)}$, it is obvious that t must also become $t\sqrt{(1 - u^2)}$.

The problem is a little more difficult when we deal with the increase of mass, but the following considerations will show why there should be an increase of mass with increase of velocity.

The velocity of a body increases indefinitely, up to a point, when a force acts continuously on it. We use the word "force" for lack of a better word, because it is a mere mathematical convention, as will appear later. There is a limit to the velocity of a body and that limit is the velocity of light. Assuming, then, that it is impossible for a body to attain the velocity of light, there must be something opposing its increase of velocity, and that something is the increasing resistance that it offers—in other words—its increase in mass.

It would be simpler in certain ways if we altered our definition of energy and took it as $mc^2/\sqrt{(1 - u^2)}$ because this expression gives a better measure of energy than the usual formula. When u is small, as it usually is in terrestrial phenomena, the above expression gives the energy as $mc^2(1 + \frac{1}{2}u^2)$ but this is not correct when we deal with high velocities such as often occur with electrons. The symbol m refers to the mass of a body when it is at rest relative to the observer, and if $u = 0$, the energy of the body becomes mc^2, that is, its mass multipled by the square of the velocity of light; this has been called the "energy of constitution" of the body. It has been shown that mass and energy can be identified, so that absorption of energy, say by heating a body, implies an increase in its mass, though this is relatively so small that it is difficult to detect it. On the other hand, parting with energy implies a decrease in mass, and this has been verified when four hydrogen atoms form a helium atom. If the hydrogen atoms could be arranged without the transformation of any material weight into radiation, the helium atom would be exactly four times the mass of the hydrogen atom. The actual ratio of the masses is only 3·970 to 4, the difference representing the energy emitted in radiation.

Relativity has made it impossible to reduce Nature to mere matter and motion, and some believe that it has dealt a very serious blow to the materialism of the last century. It is outside the scope of this work to deal with this particular aspect of the question, and we shall proceed to examine the relation of "events" to matter, space, and time. This is necessary, because the word "event" has been used on various occasions without explaining fully what it means, and readers may have found the use of the word a little misleading.

An Event

Sir Arthur Eddington* defines "event" as follows: "An event in its customary meaning would be the physical happening which occurs at and identifies a particular time and place." This is its customary meaning, but he uses the word in another sense also, which is explained in the same chapter. A point in space-time, which is the same as a given instant at a given place, is called an "event". It will assist if a specific illustration is used to explain the term.

However great the intelligence of a human being, his knowledge of nature is derived from experience. This experience is gained gradually in life from the observation of phenomena, and the process is so slow that we are not always aware of the progress that we make. It is possible to imagine a human being suddenly introduced into the phenomenal world, possessing powers of observation and of ratiocination, but devoid of previous experience. What would he perceive and what would be his interpretation of the occurrences?

Professor H. Dingle† gives a very fine description of the experience of the intelligent human being in such circumstances, and an epitome of this follows, the human being being denoted by *A*.

Event 1. *A* sees a wasp alight on an object.

Event 2. *A* sees the wasp alight on his hand.

A then begins to use his intelligence and to impose some order on the circumstances in which he finds himself.

He notices that there is something common to the two events—in particular that there is an "object" with black and yellow bands and this object characterizes the series of events between 1 and 2. He has now gained a perception of *matter* in the form of a wasp.

This, however, is not sufficient, and he must construct some other relation between the events. He does so by saying that the events are in different places—the object on which the wasp rested and his hand are in different places, and so he forms an idea of place, and by extending the same relation to other events which he perceives, he becomes conscious of "Infinite space". Matter and space have thus arisen as conceptions derived from a common source—the events themselves.

Event 3. The wasp stings *A*.

How can he relate the unpleasant sensation to event 2, the wasp alights on his hand? He finds a third type of relation and says that one of the events occurred before the other. By generalizing this relation he forms the conception of "time".

According to the relativist, then, matter, space, and time are types of relations between events, and together they appear to be capable of relating the whole of inanimate Nature in a consistent and orderly way.

* *Space, Time and Gravitation*, p. 45. † *In Relativity for All.*

A and his descendants ultimately come to regard matter, space, and time as the fundamental perceptions of the human mind, ignoring the event which sinks into insignificance. But the derivative character of matter, space and time lies at the heart of the modern principle of relativity, and the event is the immediate entity of perception. Since events finally constitute the external physical world, two observers of Nature see the same events, but not necessarily the same matter. The spatial, temporal, and material relations imposed by observers on the events will not necessarily be the same.

Chapter VII

GENERAL RELATIVITY

Up to the present we have limited ourselves to a restricted class of observers—those who are moving relatively to events with *uniform* velocity. We have seen that each observer forms his own opinion about length, time, mass and velocity, and that there is no reason why special preference should be given to the opinion of one more than of another. Now suppose observers and events move with *variable* velocity with reference to one another, what modifications, if any, will be introduced into our equations? The investigation of this subject forms the subject of General Relativity.

Imagine a lift ascending or descending with uniform velocity and that a passenger with a spring balance weighs himself when lift and passengers are ascending and then when they are descending. Those who have an elementary knowledge of dynamics know that the machine will record his exact weight on each occasion, the uniform velocity making no difference. Now suppose that the lift is ascending with an acceleration of 10 feet a second per second, or, in other words, that it is moving with increasing velocity, the velocity being augmented 10 feet a second each second of its motion. In this case a passenger who weighed 11 stones would find that the balance indicated $11\ (32 + 10)/32 = 14\frac{1}{2}$ stones approximately, 32 being the value of g where the experiment is performed. If the lift is descending with the same acceleration, that is, 10 feet a second per second, his apparent weight will be $11\ (32 - 10)/32 = 7\frac{1}{2}$ stones. If the lift is descending with acceleration 32 the passenger's weight is $11\ (32 - 32)/32$ which is 0, and in this case, which implies that the lift is falling freely, no pressure is exerted by anyone on the floor of the lift. All this is elementary and does not require further explanation.

We shall now describe an experiment which could be partly carried out, but as no one would survive to tell us about his experiences, we must accept the following without inviting anyone to verify it.

The Principle of Equivalence

Imagine a lift with a transparent bottom through which an inmate can see clearly, and imagine further that the lift with its inmate, whom we will call *I*, is taken up to a height of about 5 miles in an aeroplane and

then dropped. Ignoring atmospheric resistance and also the slight variation in gravity owing to the varying distance of the lift from the earth's centre, both lift and I will descend with an acceleration of 32 feet a second per second. The following are some of the experiences of I.

If he places anything against the walls of his temporary home or his "universe", if we may use this expression, it will remain there, because it shares the acceleration with I and the lift. If he can raise himself from the floor he will remain poised in the air between floor and ceiling. If he throws an object across his home the object will describe a straight line. All this is from I's point of view.

Now imagine an observer O on the surface of the earth who is looking at the lift and I and who, we may assume, can see through the transparent floor so that objects inside the lift are easily seen. The following are O's opinions of what takes place.

I and the lift are falling towards the earth with an acceleration equal to g at the place. Objects which I thinks are at rest inside his lift are sharing in the acceleration. An object thrown by I is not pursuing a straight line but is following a path which O knows is a parabola.

I is quite unaware of a gravitational field in his neighbourhood and if he looks through the floor of his home he will imagine that O is approaching him with an acceleration of 32 feet a second per second. If he forms any opinions of the cause of this acceleration he will conclude that O is in a field of force. We can imagine that I is able to look through the earth and see an aeroplane at O's antipodes, and if this aeroplane should crash, I will conclude that it is in another field of force of greater intensity than that which O experiences. A parachutist descending slowly in the vicinity of I's home would appear to be ascending, so I would naturally conclude that another field of force existed in the parachutist's world, but this would appear to be of less intensity than that in O's world. If the parachutist throws an object horizontally from his parachute, I sees it describing a curve which, however, differs from the curve that O sees. It is unnecessary to multiply instances of the appearances of different worlds to I. We must now enquire why I has caused so much confusion, judging from O's point of view, by creating different fields of force.

When we judge the motion of an object and ascertain its velocity, we must start with some reference point, or axes of reference, as it is generally described. For instance, if we are driving a car and another car passes us, we can ascertain its velocity relative to our car at the time, and this may be 10 miles an hour. If, however, we ascertained its velocity with reference to a point on the road, we might find its velocity to be 40 miles an hour. By selecting our axes in our car we make the other car appear to be moving much more slowly than we do when we select our axes on the road. If we were meeting the car, the

speed of each being the same as before, and we took our axes of reference in our car again, we should ascribe a speed of 70 miles an hour to the other car. (The speeds of the cars relative to the road are 40 and 30 miles an hour.)

Let us apply this reasoning to *I* and *O*.

I selected his axes in his world, and there is no reason why he should not do so, just as we are entitled to select our axes in our moving car. The fact that we are aware of the motion of our car and that *I* is unaware of the motion of his world need not concern us. What has *I* done by selecting his axes of reference in his world? He has made *O* appear to be moving towards him with an acceleration, an aviator falling to the earth at the antipodes to be also moving towards him, but with greater acceleration, and a parachutist to be moving with less acceleration. From *I*'s point of view the choice of axes was probably the most sensible thing he could do, but from the point of view of *O* he introduced considerable complications into his universe by producing artificial fields of force.

Einstein's Principle of Equivalence can now be enunciated and its meaning will be clearer after the above remarks on the choice of axes. This principle is as follows:

A gravitational field of force is precisely equivalent to an artificial field of force, so that in any small region it is impossible by any conceivable experiment to distinguish between them.

By a choice of axes *I* has neutralized in his immediate neighbourhood what *O* calls a gravitational field, but in doing so he has created a gravitational field in the neighbourhood of *O* and also of others. Although we generally consider the presence of matter responsible for creating a gravitational field, nevertheless any observer can so choose his axes that in his immediate neighbourhood all gravitational effects are neutralized.

Observer on a Rotating Disc

Let us now consider the world-line of *I* moving through a space-time domain. At each point in space-time he neutralizes the gravitational field in his immediate neighbourhood, and he can measure the separation of two close events in his career by means of his own clock. This separation will be the proper time (see p. 187) as recorded by his clock. The total separation between two events in his career, measured along his world-line, can be found, and, as we have seen, this world-line will appear straight to the observer who is moving with it, when the body is moving freely. But observers outside *I* say that the geodesic is a curve and not a straight line, because the presence of matter has distorted space-time in its neighbourhood. This will be clearer from another illustration taken from Einstein's *Relativity, The Special and the General Theory*, which is amended to a certain extent and made a little simpler for the reader.

Instead of dealing with I in a lift we shall imagine that he lives on a large disc (Fig. II (10)) rotating about an axis perpendicular to its plane. An observer O, not on the disc, says that it is rotating about this axis, but I regards the disc as motionless and imagines that O is moving in a circle in the reverse direction. I uses axes through C as axes of reference, so that any point is defined by its distances from these axes, and there is no reason why he should not do so. The case is analogous to the lift in which, as we saw, I formed his own system of reference. Let us now see how I and O will regard their experiences.

While the disc is rotating there will be a tendency for I to be moved from a point P towards the periphery, and the force causing this is proportional to the distance of P from C. At C itself there is no force, but at other points I is convinced that there is a gravitational field acting

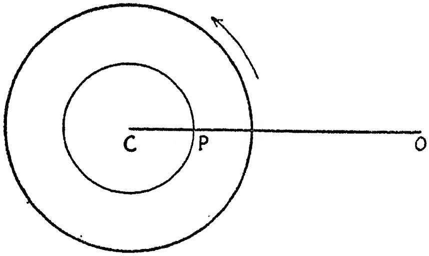

Fig. II (10)
The world of an observer on a rotating disc.

from C towards the periphery of the disc. If a body starts from C and moves in the direction CO with uniform velocity, I, who says his disc is at rest, describes the body as travelling from C along the line CO which is rotating in the direction shown by the arrow. Using his axes through C to trace the path of the body, he will say that it describes a spiral curve. Now if a body describes a spiral curve we usually attribute its motion to some force, and I naturally concludes that the spiral curve which it describes is due to a gravitational field, which, as we have seen, he believes exists. How does O regard the situation?

O says that I is moving round C in a circle with uniform speed and, just as a stone whirling round the end of a string is held in a circular path by the pull of the string, so O thinks I has an acceleration towards C which is produced by I clinging on to the disc to avoid ejection. The body which I thought moved in a spiral curve due to a field of force is described by O as moving in a straight line devoid of a field of force.

Now let us deal with measurements on the disc. Imagine that I uses his rule to find the diameter and circumference of the rotating disc. When he measures the diameter, which we may take to be 100,000 units, O will agree with his result because the rule has no velocity in the

direction of its length when a measurement is made radially. When I places the rule tangentially to the disc to measure its circumference, the rule has a velocity in the direction of its length relative to O, but not of course to I, and O says that it contracts (see p. 175), and owing to this contraction more than 314,159 measures of the rule will be necessary, according to O, to measure completely round the disc.

It is presumed that O knows that π, the ratio of the circumference of a circle to its diameter is 3·14159 approximately, and he informs I about the discrepancy. I and O agree on the number of times the rule is applied to go round the disc, and I, who is unaware of the contraction because he is moving with the rule, concludes that the ratio of the circumference of a circle to its diameter is not 3·14159 but exceeds this. For this reason I's geometry is not the geometry of Euclid, and we describe his space as non-Euclidean. It is obvious that if the speed of rotation of the disc is different or if the disc is larger or smaller, different values for π will be found by I, because the contraction-ratio varies and hence the number of times the rule must be used, according to O. Variations in the speed of rotation imply variations also in I's "gravitational fields", so that π depends on the strength of the gravitational field in I's world.*

It has been shown that clocks run at different rates according to the "contraction" of a body, which in turn depends on the velocity of the observer relative to the body (see p. 176). Hence clocks on different parts of the disc where the linear velocities are not the same, increasing from C outwards, do not run at the same rate, according to O. The greater the distance of a clock from C the slower it runs by O's reckoning, and so there is an irregularity of time-measurement as well as of space-measurement. In fact, the space-time world of I is distorted in respect of time and space.

O does not share with I the view that there is this distortion, and O considers that both space and time are uniform. The non-Euclidean character of I's space and the irregularity of time are due to the fact that I created a gravitational field by his choice of axes.

We have seen that the separation between two events was established on the assumption that space-time is uniform (pp. 192–4), but if there is a distortion of space-time this uniformity no longer exists, and if the separation is to remain the same to all observers, we must adopt a new geometry. We have generally assumed that the geometry of Euclid was the only one applicable to our universe, but it has been shown that other equally consistent geometries exist. The sum of the three angles of a triangle in Euclid's geometry is equal to two right angles, but this sum is less in Hyperbolic Geometry and greater in Elliptic Geometry. It might seem possible to put the matter to the test but to

* It has been estimated that a mass of a ton placed inside a circle of five yards radius would affect π in the twenty-fourth or twenty-fifth decimal.

do so would involve using a triangle whose sides were of enormous length compared with terrestrial standards. Gauss made the attempt to determine the sum of the three angles of a triangle by using the summits of three mountains as the corners of the triangle, but experimental errors exceeded the difference between the sum of the three angles and two right angles. Our hope lies in assuming Einstein's hypothesis and then checking it with facts. If it does not explain the facts as well as the Newtonian laws then it must be modified or rejected. If, on the other hand, it explains certain phenomena which are inexplicable on the basis of the Newtonian laws, there is a very strong presumption that it is a more accurate description of the Universe than we can obtain from Newton's laws. We shall come to the experimental verification of Einstein's hypothesis later.

How does Einstein's theory explain the movements of the heavenly bodies—for instance, the revolution of the planets round the sun? Newton explained them by the universal law of gravitation, every body in the universe attracting every other body with a force that is proportional to the product of the masses of the bodies and inversely proportional to the square of their distance apart, but now it is unnecessary to postulate the existence of this "force". It will assist at this stage if we refer to some points dealt with in the earlier portion of Part II.

Path Chosen by a Body

It has been shown on p. 184 that the separation between two events is constant for all observers, and that this separation is obtained by taking the square root of a quantity derived from the time- and space-intervals for each observer. The separation between two events in the life of a body is equal to the proper time for that body—that is, the time-interval measured by a clock in the body's universe. A body chooses the path which, in its own view, gives the greatest length of life—a rule of conduct called the "Law of Cosmic Laziness" by Bertrand Russell.*

Now take the case of the earth in its revolution round the sun from January 1 to March 1, say. Why does it move in an ellipse, not in a circle or straight line, as seen from the sun? If it moved in either of these paths the separation would have the same value for all observers though a different value from that which it has. If, therefore, we can settle something about this interval, we can formulate a statement which may be called a law of Nature.

Einstein assumed that Nature was such that the interval between any two events was a maximum. If, therefore, the earth moved in any other path different from its present path, the total four-dimensional interval between the dates selected would be smaller than it actually is.

* In *The A B C of Relativity*, p. 124.

Although there is an essential difference between Newton's assumptions and those of Einstein, deductions based on either view agree with very great accuracy, except for a few crucial cases. Newton assumed that matter, if free to move, would take the minimum spatial distance between two points on its path, or, in other words, it would move in a straight line. Einstein assumed that an event would be separated from another event by the maximum four-dimensional distance. Fortunately it has been possible to test each theory in its application to the motions of bodies, and as a result it has been shown that the actual path does not give the maximum four-dimensional interval when the geometry of Euclid is used.

On first appearance this seems fatal to Einstein's hypothesis, but there is another assumption which saves the situation for the relativity theory—the assumption that space is non-Euclidean. Of course if experiment could prove that space was Euclidean then Einstein's theory would necessarily be modified or discarded, but experiment in the ordinary way is unable to settle the matter for reasons already given. Although ordinary experiments cannot decide in favour of either hypothesis, certain very refined experiments have been made and these are entirely in favour of Einstein's theory.

Why, then, does a planet or a satellite pursue the course that it does and no other? To answer this question we shall use an analogy employed by Bertrand Russell,* and it is hoped that this will make the subject a little clearer.

Although we can make our space Euclidean in any small region in the neighbourhood of matter, we cannot do so throughout any region within which gravitation varies sensibly. If we assumed that a large region of space in the neighbourhood of matter was Euclidean we should be obliged to discard the view that bodies move in geodesics, and we wish to retain this view. In the neighbourhood of matter there is a hill in space-time, using an analogy which must not be taken too seriously, and this hill grows steeper as it gets nearer the top, ending in a sheer precipice. By the law of cosmic laziness a body will not attempt to go straight over the hill but will go round it. The body does not do this because of any attraction exercised on it by the larger body nor because of any mysterious "force"; it follows this path simply because of the nature of space-time in its vicinity. Hence, instead of dealing with the motions of bodies—planets and others—by dynamical equations, the problem is merely one in geometry.

Mr. Russell's analogy to clarify this point is very helpful. He asks us to imagine a number of people walking across a great plain on a dark night, one part of the plain containing a great hill with a flaring-beacon light on the top. The hill is supposed to get steeper as we ascend and finally to end in a precipice. Villages are dotted about the plain, and

* *Ibid.*, pp. 127–9.

men carrying lanterns are walking from village to village, paths having been made to show the easiest way. To avoid going up the hill these paths will be more or less curved, and near the top of the hill they will be more sharply curved than they are lower down. An observer from a balloon, knowing nothing about the hill and unable to see the ground by night, will observe people turning out of a straight course when they approach the beacon, and they will turn aside still more as they come closer to it. The observer, who has no previous knowledge about the configuration of the country, will conclude that the movements of the people in various curves are due to an effect of the beacon—perhaps it is very hot and people avoid it for fear of being burned. If the balloonist waits for daylight he will see that the beacon merely marks the top of the hill and exercises no influence on the people with their lanterns.

In this analogy the beacon corresponds to the sun, the people with lanterns to planets and comets, the paths correspond to their orbits, and the coming of daylight to the coming of Einstein, who says that the sun is at the top of a hill in space-time. Each body at each moment adopts the course easiest for it, but owing to the hill this course is not a straight line. Every body pursues the easiest course from place to place, but this course is affected by the hills and valleys that are encountered on the way. If we walk through a wood the most speedy course from one end to the other is not always a straight line; owing to the obstruction of trees and undergrowth it may be necessary to make a detour in many cases and we shall reach the other end sooner than we could do by following a straight line.

Although Einstein's law of gravitation gives practically the same results as Newton's when applied to the computation of the orbits of comets, planets, satellites, etc., there are a few cases in which Einstein's law is better than Newton's. Einstein published his views on special relativity in 1905 and on general relativity in 1915, and he pointed out that the peculiar motion of the perihelion of the orbit of the planet Mercury, which had puzzled astronomers for many years, could be accounted for by his general relativity.

Verification of Einstein's Theory

All the planets, Mercury included, move round the sun in ellipses, so it may seem remarkable that Mercury was selected out of all the planets to verify the Einstein hypothesis. The reason was because Mercury moves in a very eccentric orbit and, in addition, being the closest planet to the sun, has a higher orbital velocity than any other planet. At one time the planet comes within 28½ million miles from the sun, and at another time, 44 days later, it is over 43 million miles from the sun, its velocities on these occasions being 33 and 27 miles a second respectively. Mercury is disturbed slightly by the other planets, being

pulled a little out of its course, and in consequence its nearest position to the sun, that is, its perihelion, is not the same from year to year. In fact it has to move through a little more than 360° at each revolution to return to its nearest point to the sun. Now astronomers are able to compute the amount of disturbance or perturbation, as it is called, which Mercury suffers from the other planets, and so they were able to explain the movement of its perihelion, but not exactly. There was a discrepancy of 43 seconds of arc per century—a very small amount, it is true—and astronomers were very puzzled about it because no known facts about the solar system would explain it. It was believed by some that there was a small planet between Mercury and the sun which had escaped detection, and its mass and distance from the sun were calculated to fit in with the extra 43 seconds of arc, this planet being supposed to produce additional perturbations. Search was made for "Vulcan", as this hypothetical planet was called, but it was never found and indeed never will be because it does not exist. Asaph Hall attempted to solve the problem by assuming that Newton's inverse square law did not hold exactly, the attraction between two bodies varying as $1/r^{2(1+d)}$, where d is only 1/13,000,000. This new "law" would explain the discrepancy so far as Mercury was concerned but introduced a discrepancy in the nearest position of the moon to the earth, known as the moon's perigee.

Einstein explained the discrepancy very easily and did not introduce complications in other phenomena by doing so—on the contrary, he explained other phenomena that were inexplicable on the Newtonian laws. A simple explanation of the behaviour of Mercury under the Einstein hypothesis is as follows.

From what has been said about mass and velocity we can surmise that the force of attraction (using the Newtonian expression for the present) increases with the speed of the body and vice versa. When Mercury is at its greatest distance from the sun the slight defect in the force implies a longer time to return to perihelion. When Mercury is at perihelion the excess of the force means that the planet takes a longer time to reach aphelion—its greatest distance from the sun—and in each case perihelion moves forward. The formula for computing the amount of this movement is given in the Appendix, and the reader will see, if he applies it to the other planets, that the amount is too small to be detected. A fairly high eccentricity of the orbit is necessary for detection; if Mercury had not combined the two qualities of moving in a highly elliptic orbit (high at least for a planet), and also of moving in an orbit comparatively close to the sun, the discrepancy in the motion of its perihelion might never have been discovered.

According to the general theory of relativity, a ray of light will experience a curvature of its path when it is passing through a gravitational field. This curvature is similar to that experienced by the path of a body which is projected in a gravitational field. According to

Newton's laws also there should be a curvature of the path of the ray, but calculations showed that the deflection in this latter case should be only one half of what it should be under the relativity theory. It is not easy to submit the matter to a crucial test because it is not often that a star almost in line with the sun can be seen. It can be seen during a total eclipse provided the star is sufficiently bright, but stars sufficiently bright are not always in the correct position during total solar eclipses. Fortunately on May 29, 1919, the sun was close to some bright stars during a total eclipse, and the Royal Society and the Royal Astronomical Society equipped two expeditions to obtain photographs, one to Sobral, in Brazil, and the other to Principe, West Africa. Unfortunately clouds interfered badly with the expedition to Principe, but conditions were excellent at Sobral. The party at the latter station remained for two months after the eclipse to photograph the same region of the sky before dawn, so that they might have comparison photographs taken under the same conditions. It is remarkable that some of the photographs taken at Sobral pointed to agreement with the Newtonian value, but certain complications diminished the value of these plates. A set of seven plates taken at Sobral, the measurements of which had been delayed for certain reasons, provided the final decision, and their verdict was indisputably in favour of Einstein's value for the deflection. The story is fully described by Sir Arthur Eddington in *Space, Time and Gravitation*, Chapter VII, and his conclusion of the matter is summarized as follows:

"Those who regard Einstein's law of gravitation as a natural deduction from a theory based on the minimum of hypotheses will be satisfied to find that his remarkable prediction is quantitatively confirmed by observation, and that no unforeseen cause has appeared to invalidate the test."

Similar tests at subsequent eclipses have corroborated those of the 1919 eclipse, and the matter is now regarded as established beyond any possibility of doubt.

The vibration of an atom can be regarded as providing us with a natural clock, and if we measure the separation between the beginning and end of a vibration in two atoms which are identical, the result should be the same, other circumstances being identical. If one of the atoms is close to a massive body—say the sun—it can be shown that its period of vibration is slightly slower than the period of the same atom removed from the neighbourhood of a massive body or on a body less massive than the sun. As a consequence the solar atom would be expected to vibrate slower than the atom on the earth and a small shift in the solar spectrum towards the red should take place, when the spectrum is compared with that of the same atom on the earth. Although this shift is very small in the case of the sun, its effect is more noticeable when the physicist deals with the stars of very great density—the white dwarfs—such as the companion of Sirius, and no doubt now remains

that the Einstein effect is in evidence in these cases. The confirmation of Einstein's theory by three independent lines of research just mentioned is a wonderful tribute to its ability to unify those laws which have won a place in human knowledge held today by physical science.

Although Relativity is a physical theory and hence is no more philosophical than any other physical theory, nevertheless it has a considerable importance for philosophers—probably more than any other branch of physics. Its chief importance for the philosopher is found in its implications regarding the character of physical thought, but it is beyond the scope of this book to enter into such questions. Readers will find them discussed in some of the more advanced works dealing with philosophy and physical science. It need scarcely be remarked that such problems are of the utmost importance and we shall refer very briefly to one of these on which there is a great diversity of opinion.

Kant thought that we ought to be able to build up a pure science of nature solely by the use of *a priori* knowledge. A similar view was held by Sir Arthur Eddington, who believed that from epistemological considerations we can foresee all the laws of nature that are generally classified as fundamental. (Epistemology is the science of knowledge.) On this view an intelligence unacquainted with our universe but acquainted with the system of thought by which the mind is able to interpret to itself the content of its sensory experience, would be able to attain to all the knowledge of physics that has been attained by experiment. His view does not imply that the particular objects and events of our experience could be deduced *a priori*, but the generalizations that we have based on them could be deduced in this way.

It is very difficult to accept this view, and most physicists reject it, although Sir Arthur Eddington's arguments are not always easy to confute. The subject is merely referred to here to show that philosophical problems arise which touch on the province of the physicist, but a discussion of these does not lie within the scope of this work.

APPENDIX I

Composition of Velocities

Let a ball move along a groove cut in a board in the direction OA', and at the same time let the board be moved along the table on which it rests in the direction OB'. What is the velocity of the ball with reference to the table?

This problem is solved by laying off OA to represent the magnitude of the velocity of the ball in the groove, using any convenient scale, say an inch or a centimetre to represent a velocity of one foot per second, and

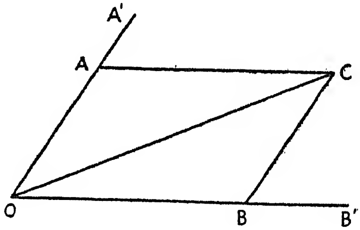

Fig. II (11)
The composition of velocities.

OB, on the same scale, to represent the velocity of the board. Complete the parallelogram $OACB$ and draw the diagonal OC. This diagonal will represent in magnitude and direction the velocity of the ball with reference to the table, on the scale adopted (see Fig. II (11)).

If c is the velocity of the board and v the velocity of the ball in the groove, the magnitude of the resultant velocity V is easily obtained from the elementary properties of a triangle.

In the triangle OAC, $OC^2 = OA^2 + AC^2 - 2\,OA \cdot AC \cos \widehat{OAC}$. Now OC is V, OA is v, OB is c, and $\cos \widehat{OAC}$ is $-\cos \widehat{AOB}$, hence

$$V^2 = v^2 + c^2 + 2vc \cos \widehat{AOB}.$$

When the resultant V is given in direction and it is required to find the direction of OB with reference to OC, a graphical construction can

be used, or the direction may be computed. In the example on p. 168 the resultant OC is directed at right angles to OA, and in these circumstances $\sin \widehat{BOC} = BC/BO = OA/OB = v/c$. (Readers should draw a new figure).

In addition, $OC^2 = OB^2 - BC^2 = OB^2 - OA^2$, from which

$$V^2 = c^2 - v^2.$$

APPENDIX II

Note on the Speed of the Boats (p. 170)

Since A's speed across the stream is $\sqrt{(c^2 - v^2)}$, and the double distance is $2c$, A's journey across and back again will take a time $2d/\sqrt{(c^2-v^2)}$.

The time required by B to go down stream is $d/(c + v)$, and the time to return is $d/(c - v)$. Hence B's total time is the sum of these two expressions, which is $2cd/(c^2 - v^2)$.

The ratio of the times required by A and B to perform the double journey is $2d/\sqrt{(c^2 - v^2)}$ divided by $2cd/(c^2 - v^2)$, which is easily seen to be $\sqrt{(c^2 - v^2)}/c$.

APPENDIX III

Relation Between Time- and Distance-Intervals

It is easy to derive the expressions for s_0 and t_0 from those for s_a and t_a on p. 182.

$$\text{Let } k = \sqrt{(1 - u^2)} \text{ or } k^2 = 1 - u^2. \text{ Then since}$$
$$s_a = (s_0 - ut_0)/\sqrt{(1 - u^2)},$$

$$(s_a + ut_a)/\sqrt{(1 - u^2)} = (s_0 - ut_0 + ut_0 - u^2 s_0)/k^2 = s_0\,(1 - u^2)/k^2 = s_0,$$
$$\text{since } 1 - u^2 = k^2.$$

$$\text{Again } (t_a + us_a)/\sqrt{(1 - u^2)} = (t_0 - us_0 + us_0 - u^2 t_0)/k^2 =$$
$$t_0\,(1 - u^2)/k^2 = t_0.$$

From the above it is seen that

$$s_0 = (s_a + ut_a)/\sqrt{(1 - u^2)}, \quad t_0 = (t_a + us_a)/\sqrt{(1 - u^2)}.$$

We see from these that the relations which express A's opinion of O's world are consistent with those which express O's opinion of A's world, and in addition, that one can be deduced from the other.

APPENDIX IV

The Separation of Events

On p. 192 it was shown, from a particular case, that the separation was the same for different observers. A general proof is as follows.

Using the values of t_a and s_a on p. 182, and writing k^2 for $1 - u^2$,

$$t_a^2 - s_a^2 = (t_0^2 - 2us_0t_0 + u^2s_0^2 - s_0^2 + 2us_0t_0 - u^2t_0^2)/k^2.$$

This reduces to

$$\big(t_0^2(1 - u^2) - s^2(1 - u^2)\big)/k^2 = t_0^2 - s_0^2.$$

APPENDIX V

Centrifugal Force

The force experienced by I on the disc, p. 210, is proportional to his distance from the centre. This is easily shown by making use of the ordinary principles of a rotating body.

The centrifugal force at a distance r from the centre is v^2/r, where v is the velocity of I at this distance. If w is the angular velocity of the disc, then $v = wr$, and substituting this value of v in the above expression for the force, it becomes w^2r, which is proportional to the distance of I from the centre, provided w remains the same. If I is at the centre of the disc $r = 0$, and he will experience no force.

APPENDIX VI

Rotation of the Major Axis of a Planet's Orbit

The motion of the perihelion of a planet's orbit can be found as follows.

Let u denote the mean orbital velocity of a planet, that of light being the unit, then in each revolution of the planet its perihelion will advance $3u^2$ of a revolution. In the case of Mercury u is 16×10^{-5} approximately, and in a century there are 415 revolutions of Mercury round the sun. Hence in this period the perihelion will advance by 3187×10^{-8} of a revolution, or, since a revolution corresponds to 360° or 1,296,000″, the advance in a century will be over 41″. The

figures used are only approximate but the result is close to the actual figures, 43″.

In the case of the earth u is nearly 10^{-4} and hence $3u^2$ is 3×10^{-8}. In a century this would be 3×10^{-6} of a revolution, or nearly 3″.9, and would be observable if the earth's orbit were sufficiently eccentric. But as the earth's orbit is nearly circular (the eccentricity is about 1/60) it is impossible to be very precise about the earth's perihelion position. This will be more obvious if we think of an orbit which is circular; in this case there is no perihelion, all points on the orbit being at the same distance from the sun. The planet Venus moves in an orbit which is nearly circular, the eccentricity being only 0·0068, and so it would be impossible to use Venus to determine the Einstein effect. When we go beyond the earth's orbit we are dealing with smaller velocities of the planets and hence the effect diminishes. Mercury is the only suitable planet for testing Einstein's theory.

APPENDIX VII

Deflection of a Ray of Light by a Body

If the gravitational mass of the sun is m and if r is the mean distance of a planet from the sun, the acceleration of the planet towards the sun is denoted by m/r^2. Assuming that the mean orbital velocity of the planet is u, the acceleration radially is u^2/r, so that $m/r^2 = u^2/r$, or $m = u^2 r$. Since u is expressed in terms of the velocity of light as the unit, r must be taken in light-seconds, that is, $r = 500$. Hence in the case of the earth where u is 10^{-4}, $m = 5 \times 10^{-6}$. Since light travels 3×10^{10} cm. per second, m is 15×10^4 cm. or 1·5 kilometres.

It has been shown that the deflection of a ray of light passing at a distance r from the centre of the sun is $4m/r$ on Einstein's theory and $2m/r$ on the Newtonian theory. If we substitute 697×10^3 kilometres for r and 1·5 for m, the deflection is $6/(697 \times 10^3)$ radian or 1″.75 on Einstein's theory, or 0″.87 on the Newtonian theory. The 1919 eclipse verified the former (see p. 216).

WORKS ON RELATIVITY

Readers who desire fuller information on the subject will find the following books very helpful. Some of them are quite elementary and others very advanced; they are arranged in order, the simplest being placed first and the more difficult at the end of the list.

Sir Oliver Lodge, *Relativity*. (Methuen, 1925.)

H. Dingle, *Relativity for All*. (Methuen, 1922.)

H. Dingle, *The Special Theory of Relativity*. (Methuen, 1940.)

Bertrand Russell, *The A B C of Relativity*. (Kegan Paul, Trench, Trubner & Co., 1925.)

Albert Einstein, *Relativity, The Special and the General Theory*. (Methuen, 1920.)

Charles Nordmann, *Einstein and the Universe*. (T. Fisher Unwin, 1922.)

Edwin E. Slosson, *Easy Lessons on Relativity*. (George Routledge & Sons, Ltd., 1921.)

Moritz Schlick, *Space and Time in Contemporary Physics*. (Oxford University Press, 1920.)

Clement V. Durell, *Readable Relativity*. (G. Bell & Sons, Ltd., 1925.)

F. W. Lanchester, *Relativity*. (Constable, 1935.)

W. H. McCrea, *Relativity Physics*. (Methuen, 1935.)

Sir Arthur Eddington, *Space, Time and Gravitation*. (Cambridge University Press, 1923.)

Sir Arthur Eddington, *The Mathematical Theory of Relativity*. (Cambridge University Press, 1924.)

There are many other works on the subject, but the above cover practically all that readers will require.

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